Bergman Kernel in Convex Domains

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$\Omega \subset \mathbb{C}^n$, $w \in \Omega$

$$K_\Omega(w) = \sup\{|f(w)|^2 : f \in \mathcal{O}(\Omega),\ \int_\Omega |f|^2 d\lambda \le 1\}$$

(Bergman kernel on the diagonal)

$$\begin{aligned} G_w(z) &= G_\Omega(z,w) \\ &= \sup\{u(z) : u \in PSH^-(\Omega) : \overline{\lim_{z \to w}} \big(u(z) - \log|z-w|\big) < \infty\} \end{aligned}$$

(pluricomplex Green function)

Theorem 0 Assume Ω is pseudoconvex in $\mathbb{C}^n$. Then for $w \in \Omega$ and $t \le 0$

$$K_\Omega(w) \ge \frac{1}{e^{-2nt}\lambda(\{G_w < t\})}.$$

Optimal constant: "=" if $\Omega = B(w,r)$.

Proof 1 Using Donnelly-Fefferman's estimate for $\bar\partial$ one can prove

$$K_\Omega(w) \geq \frac{1}{c(n,t)\lambda(\{G_w < t\})}, \tag{1}$$

where

$$c(n,t) = \left(1 + \frac{C}{Ei(-nt)}\right)^2, \quad Ei(a) = \int_a^\infty \frac{ds}{se^s}$$

(B. 2005). Now use the tensor power trick: $\widetilde{\Omega} = \Omega \times \cdots \times \Omega \subset \mathbb{C}^{nm}$, $\widetilde{w} = (w, \ldots, w)$ for $m \gg 0$. Then

$$K_{\widetilde{\Omega}}(\widetilde{w}) = (K_\Omega(w))^m, \quad \lambda(\{G_{\widetilde{w}} < t\}) = (\lambda(\{G_w < t\}))^m,$$

and by (1) for $\widetilde{\Omega}$

$$K_\Omega(w) \geq \frac{1}{c(nm,t)^{1/m}\lambda(\{G_w < t\})}.$$

But $\lim\limits_{m\to\infty} c(nm,t)^{1/m} = e^{-2nt}$. $\square$

Proof 2 (Lempert) By Berndtsson's result on log-(pluri)subharmonicity of the Bergman kernel for sections of a pseudoconvex domain it follows that $\log K_{\{G_w<t\}}(w)$ is convex for $t \in (-\infty, 0]$. Therefore

$$t \longmapsto 2nt + \log K_{\{G_w<t\}}(w)$$

is convex and bounded, hence non-decreasing. It follows that

$$K_\Omega(w) \geq e^{2nt} K_{\{G_w<t\}}(w) \geq \frac{e^{2nt}}{\lambda(\{G_w < t\})}. \quad \square$$

Berndtsson: This method can be improved to show the Ohsawa-Takegoshi extension theorem with optimal constant.

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$$K_\Omega(w) \geq \frac{1}{e^{-2nt}\lambda(\{G_w < t\})}.$$

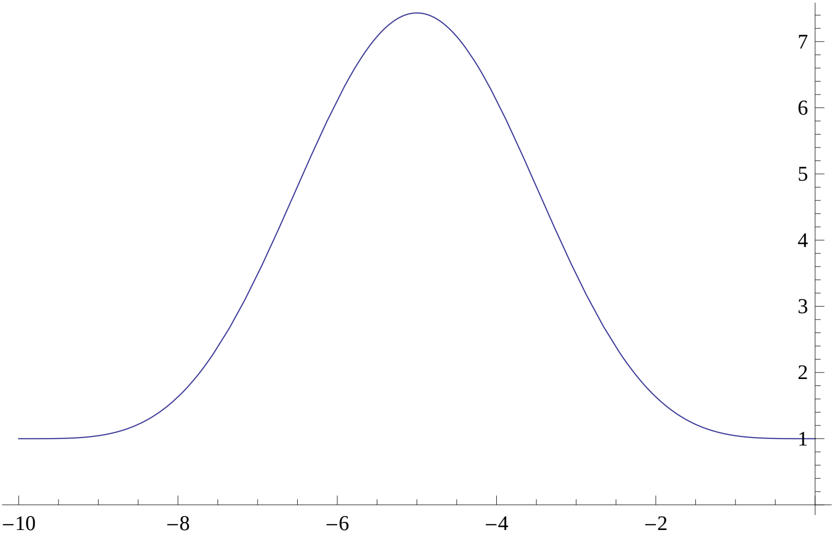
What happens when $t \to -\infty$? For $n = 1$ Theorem 0 immediately gives:

Theorem (Suita conjecture) For a domain $\Omega \subset \mathbb{C}$ one has

$$K_\Omega(w) \geq c_\Omega(w)^2/\pi, \quad w \in \Omega, \tag{2}$$

where $c_\Omega(w) = \exp\left(\lim_{z\to w}(G_\Omega(z,w) - log|z-w|)\right)$
(logarithmic capacity of $\mathbb{C} \setminus \Omega$ w.r.t. w).

Theorem (Guan-Zhou) Equality holds in (2) iff $\Omega \simeq \Delta \setminus F$, where Δ is the unit disk and F a closed polar subset.



$\frac{\pi K_\Omega}{c_\Omega^2}$ for $\Omega = \{e^{-5} < |z| < 1\}$ as a function of $2\log|w|$

What happens with $e^{-2nt}\lambda(\{G_w < t\})$ as $t \to -\infty$ for arbitrary n? For convex Ω using Lempert's theory one can get

Proposition If Ω is bounded, smooth and strongly convex in $\mathbb{C}^n$ then for $w \in \Omega$

$$\lim_{t\to-\infty} e^{-2nt}\lambda(\{G_w < t\}) = \lambda(I_\Omega^K(w)),$$

where $I_\Omega^K(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}$ (Kobayashi indicatrix).

Corollary If $\Omega \subset \mathbb{C}^n$ is convex then

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega^K(w))}, \quad w \in \Omega.$$

For general Ω one can prove

Theorem (B.-Zwonek) If Ω is bounded and hyperconvex in $\mathbb{C}^n$ and $w \in \Omega$ then

$$\lim_{t\to-\infty} e^{-2nt}\lambda(\{G_w < t\}) = \lambda(I_\Omega^A(w)),$$

where $I_\Omega^A(w) = \{X \in \mathbb{C}^n : \overline{\lim}_{\zeta\to 0}\left(G_w(w+\zeta X) - \log|\zeta|\right) \leq 0\}$

(Azukawa indicatrix)

Corollary (SCV version of the Suita conjecture) If $\Omega \subset \mathbb{C}^n$ is pseudoconvex and $w \in \Omega$ then

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega^A(w))}.$$

Conjecture 1 For Ω pseudoconvex and $w \in \Omega$ the function

$$t \longmapsto e^{-2nt}\lambda(\{G_w < t\})$$

is non-decreasing in t.

It would easily follow from the following:

Conjecture 2 For Ω pseudoconvex and $w \in \Omega$ the function

$$t \longmapsto \log \lambda(\{G_w < t\})$$

is convex on $(-\infty, 0]$.

Theorem (B.-Zwonek) Conjecture 1 is true for $n = 1$.

Proof It is be enough to prove that $f'(t) \geq 0$ where

$$f(t) := \log \lambda(\{G_w < t\}) - 2t$$

and t is a regular value of G_w. By the co-area formula

$$\lambda(\{G_w < t\}) = \int_{-\infty}^{t} \int_{\{G_w = s\}} \frac{d\sigma}{|\nabla G_w|} ds$$

and therefore

$$f'(t) = \frac{\displaystyle\int_{\{G_w = t\}} \frac{d\sigma}{|\nabla G_w|}}{\lambda(\{G_w < t\})} - 2.$$

By the Schwarz inequality

$$\int_{\{G_w = t\}} \frac{d\sigma}{|\nabla G_w|} \geq \frac{(\sigma(\{G_w = t\}))^2}{\displaystyle\int_{\{G_w = t\}} |\nabla G_w| d\sigma} = \frac{(\sigma(\{G_w = t\}))^2}{2\pi}.$$

The isoperimetric inequality gives

$$(\sigma(\{G_w = t\}))^2 \geq 4\pi\lambda(\{G_w < t\})$$

and we obtain $f'(t) \geq 0$. $\square$

Conjecture 1 for arbitrary n is equivalent to the following *pluricomplex isoperimetric inequality* for smooth strongly pseudoconvex Ω

$$\int_{\partial\Omega} \frac{d\sigma}{|\nabla G_w|} \geq 4n\pi\lambda(\Omega).$$

Conjecture 1 also turns out to be closely related to the problem of symmetrization of the complex Monge-Ampère equation.

What about corresponding upper bound in the Suita conjecture?
Not true in general:

Proposition (B.-Zwonek) Let $\Omega = \{r < |z| < 1\}$. Then

$$\frac{K_\Omega(\sqrt{r})}{(c_\Omega(\sqrt{r}))^2} \geq \frac{-2\log r}{\pi^3}.$$

It would be interesting to find un upper bound of the Bergman kernel for domains in $\mathbb{C}$ in terms of logarithmic capacity which would in particular imply the $\Rightarrow$ part in the well known equivalence

$$K_\Omega > 0 \;\Leftrightarrow\; c_\Omega > 0$$

($c_\Omega^2 \leq \pi K_\Omega$ being a quantitative version of $\Leftarrow$).

The upper bound for the Bergman kernel holds for convex domains:

Theorem (B.-Zwonek) For a convex Ω and $w \in \Omega$ set

$$F_\Omega(w) := \big(K_\Omega(w)\lambda(I_\Omega^K(w))\big)^{1/n}.$$

Then $F_\Omega(w) \leq 4$. If Ω is in addition symmetric w.r.t. w then $F_\Omega(w) \leq 16/\pi^2 = 1.621\ldots$.

Sketch of proof Denote $I := int\, I_\Omega^K(w)$ and assume that $w = 0$. One can show that $I \subset 2\,\Omega$ ($I \subset 4/\pi\,\Omega$ if Ω is symmetric). Then

$$K_\Omega(0)\lambda(I) \leq K_{I/2}(0)\lambda(I) = \frac{\lambda(I)}{\lambda(I/2)} = 4^n. \qquad \square$$

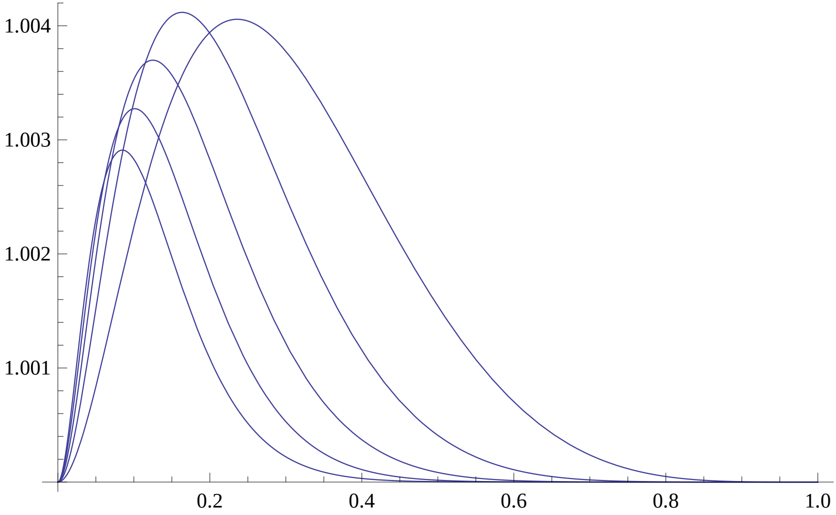
For convex domains F_Ω is a biholomorphically invariant function satisfying $1 \le F_\Omega \le 4$. Can we find an example with $F_\Omega(w) > 1$? Using Jarnicki-Pflug-Zeinstra's formula for geodesics in convex complex ellipsoids (which is based on Lempert's theory) one can show the following

Theorem (B.-Zwonek) Define

$$\Omega = \{z \in \mathbb{C}^n : |z_1| + \dots + |z_n| < 1\}.$$

Then for $w = (b, 0, \dots, 0)$, where $0 < b < 1$, one has

$$\begin{aligned} K_\Omega(w)\lambda(I_\Omega^K(w)) &= 1 + (1-b)^{2n} \frac{(1+b)^{2n} - (1-b)^{2n} - 4nb}{4nb(1+b)^{2n}} \\ &= 1 + \frac{(1-b)^{2n}}{(1+b)^{2n}} \sum_{j=1}^{n-1} \frac{1}{2j+1} \binom{2n-1}{2j} b^{2j}. \end{aligned}$$



$F_\Omega(b, 0, \ldots, 0)$ in $\Omega = \{|z_1| + \cdots + |z_n| < 1\}$ for $n = 2, 3, \ldots, 6$.

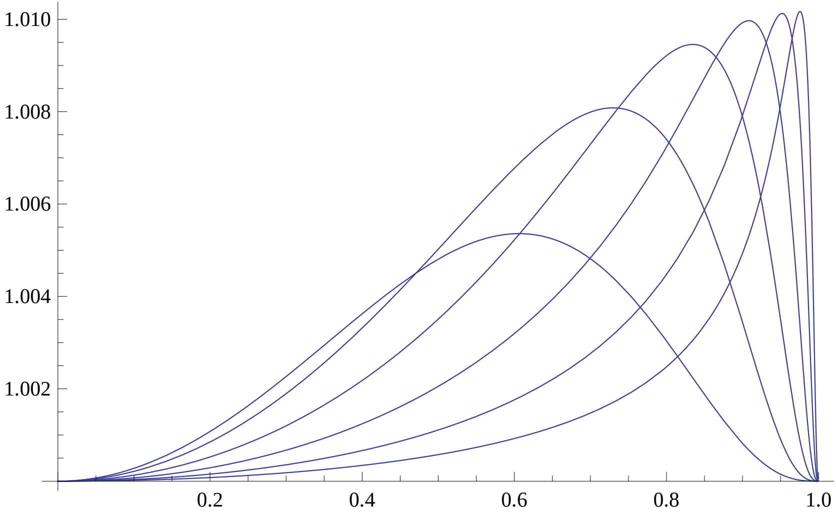
Theorem (B.-Zwonek) For $m \geq 1/2$ set $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ and $w = (b, 0)$, $0 < b < 1$. Then

$$K_\Omega(w)\lambda(I_\Omega^K(w)) = P\frac{m(1-b^2)+1+b^2}{2(1-b^2)^3(m-2)m^2(m+1)(3m-2)(3m-1)},$$

where

$$\begin{aligned} P =& b^{6m+2}\left(-m^3+2m^2+m-2\right) + b^{2m+2}\left(-27m^3+54m^2-33m+6\right) \\ &+ b^6 m^2\left(3m^2+2m-1\right) + 6b^4 m^2\left(3m^3-5m^2-4m+4\right) \\ &+ b^2\left(-36m^5+81m^4+10m^3-71m^2+32m-4\right) \\ &+ 2m^2\left(9m^3-27m^2+20m-4\right). \end{aligned}$$

In this domain all values of F_Ω are attained for $(b, 0)$, $0 < b < 1$.



$F_\Omega(b,0)$ in $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ for $m = 4, 8, 16, 32, 64, 128$.

$$\sup_{0<b<1} F_\Omega(b,0) \to 1.010182\ldots \text{ as } m \to \infty$$

What is the highest value of F_Ω for convex Ω?

What can be said the function $w \longmapsto -\log \lambda(I_\Omega^A(w))$?

Is it plurisubharmonic?

It does not have to be C^2:

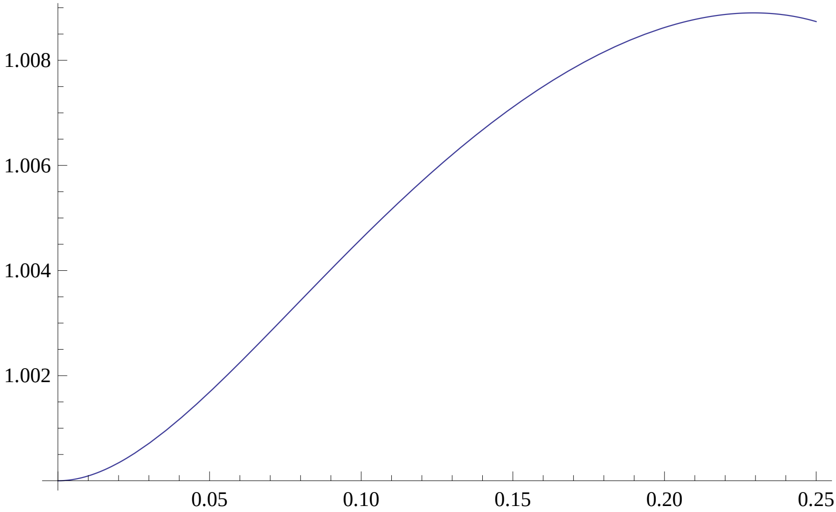
Theorem (B.-Zwonek) If $\Omega = \{|z_1| + |z_2| < 1\}$ and $0 < b \leq 1/4$,

$$\lambda(I_\Omega^K((b,b))) \\ = \frac{\pi^2}{6}\left(30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1\right).$$

$\lambda(I_\Omega^K((b,b)))$ is not C^2 at $b = 1/4$.

It is known (Hahn-Pflug) that for $0 < b < 1/2$:

$$K_\Omega((b,b)) = \frac{2\left(8b^4 - 6b^2 + 3\right)}{\pi^2(1-4b^2)^3}.$$



$F_\Omega(b, b)$ in $\Omega = \{|z_1| + |z_2| < 1\}$ for $0 < b \leq 1/4$.

Thank you!