Hörmander's Estimate, Suita Conjecture and the Ohsawa-Takegoshi Extension Theorem

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Plan of the talk

I. Suita Conjecture
II. Ohsawa-Takegoshi Extension Theorem
III. Hörmander's Estimate for $\bar{\partial}$-equation
IV. Main Result
V. ODE Problem
VI. Remaining Open Problem

I. Suita Conjecture

Green function for $D \subset \mathbb{C}$:

$$\begin{cases} \Delta G_D(\cdot, z) = 2\pi\delta_z \\ G_D(\cdot, z) = 0 \text{ on } \partial D \end{cases}$$

$c_D(z) := \exp \lim_{\zeta \to z} (G_D(\zeta, z) - \log|\zeta - z|)$

(logarithmic capacity of $\mathbb{C} \setminus D$ w.r.t. z)

$K_D(z) := \sup\{|f(z)|^2 : f \text{ holomorphic in D}, \int_D |f|^2 d\lambda \le 1\}$

(Bergman kernel on the diagonal)

Suita conjecture (1972): $c_D^2 \le \pi K_D$

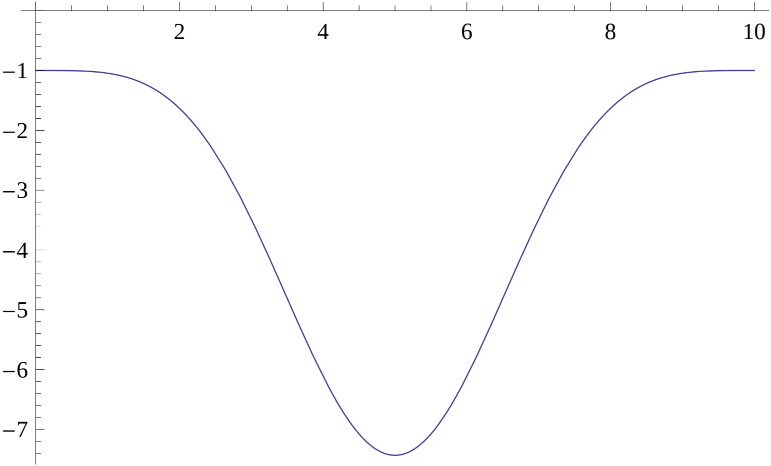
Geometric interpretation: since

$$K_D = \frac{1}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} (\log c_D)$$

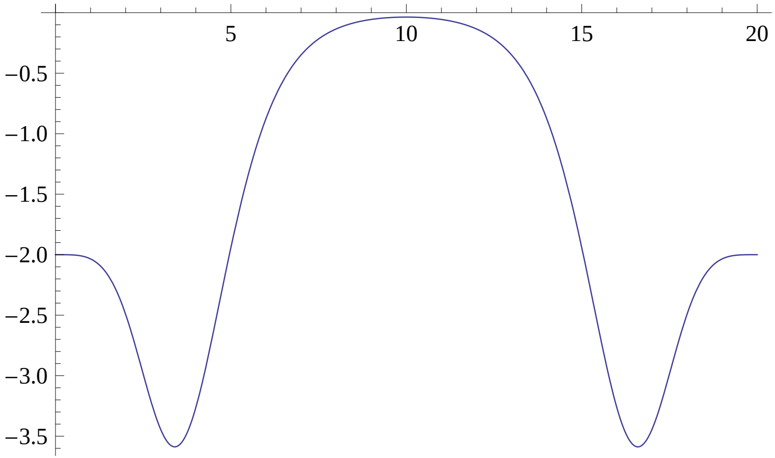
it is equivalent to

$$Curv_{c_D|dz|} \le -1$$

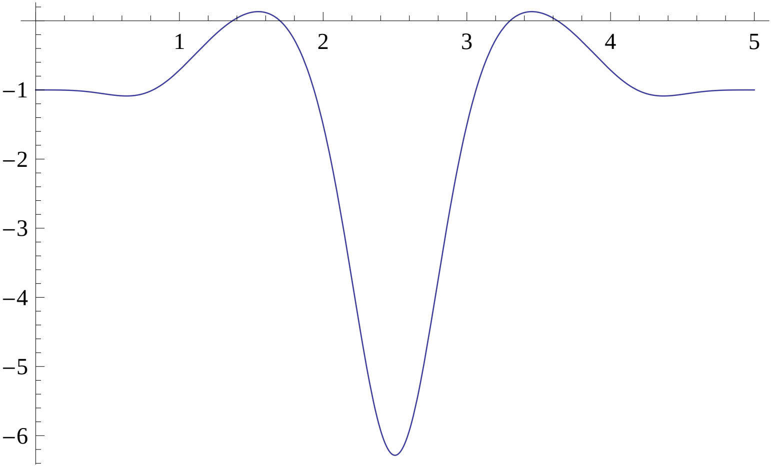
- "$=$" if D is simply connected
- "$<$" if D is an annulus and thus any regular doubly connected domain (Suita)



$Curv_{c_D|dz|}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $t = -2\log|z|$



$Curv_{K_D|dz|^2}$ for $D = \{e^{-10} < |z| < 1\}$ as a function of $t = -2\log|z|$



$Curv_{(\log K_D)_{z\bar{z}}|dz|^2}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $t = -2\log|z|$

II. Ohsawa-Takegoshi Extension Theorem

$\Omega \subset \mathbb{C}^n$, $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$

φ is called plurisubharmonic (psh) if $\left(\frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}\right) \geq 0$

Ω is called pseudoconvex if there exists smooth psh exhaustion of Ω

Theorem (Ohsawa-Takegoshi, 1987)
Ω - bounded pseudoconvex domain in $\mathbb{C}^n$, φ - psh in Ω
H - complex affine subspace of $\mathbb{C}^n$
f - holomorphic in $\Omega' := \Omega \cap H$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq C(n, \operatorname{diam} \Omega) \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

Theorem (Berndtsson, 1994)
Ω - pseudoconvex in $\mathbb{C}^{n-1} \times \{|z_n < 1\}$, φ - psh in Ω
f - holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq 4\pi \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

Ohsawa (1995) observed that the Suita conjecture is equivalent to: for $z \in D$ there exists holomorphic f in D such that $f(z) = 1$ and

$$\int_D |f|^2 d\lambda \leq \frac{\pi}{(c_D(z))^2}.$$

Using the methods of the $\bar{\partial}$-equation he showed the estimate

$$c_D^2 \leq C\pi K_D$$

with $C = 750$. This was later improved to $C = 2$ (B., 2007) and to $C = 1.954$ (Guan-Zhou-Zhu, 2011).

Theorem (Ż Dinew, 2007)
Ω - pseudoconvex in $\mathbb{C}^{n-1} \times D$, where $0 \in D \subset \mathbb{C}$, φ - psh in Ω,
f - holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{4\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

In 2011 B.-Y. Chen showed that the Ohsawa-Takegoshi extension theorem can be shown using directly Hörmander's estimate for $\bar{\partial}$-equation!

III. Hörmander's Estimate

$\alpha = \sum_j \alpha_j d\bar{z}_j \in L^2_{loc,(0,1)}(\Omega), \quad \Omega \subset \mathbb{C}^n$

Assume that $\bar\partial\alpha = 0$ (that is $\dfrac{\partial\alpha_j}{\partial\bar{z}_k} = \dfrac{\partial\alpha_k}{\partial\bar{z}_j}$)

Looking for $u \in L^2_{loc}(\Omega)$ solving $\bar\partial u = \alpha$ with estimates.

Theorem (Hörmander, 1965)
Ω - pseudoconvex in $\mathbb{C}^n$, φ - smooth, strongly psh in Ω
Then for every $\alpha \in L^2_{loc,(0,1)}(\Omega)$ with $\bar\partial\alpha = 0$ one can find $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{-\varphi} d\lambda.$$

Here $|\alpha|^2_{i\partial\bar\partial\varphi} = \sum_{j,k} \varphi_{j\bar{k}} \bar\alpha_j \alpha_k$, where $(\varphi^{j\bar{k}}) = (\partial^2\varphi/\partial z_j \partial\bar{z}_k)^{-1}$, is the length of α w.r.t. the Kähler metric $i\partial\bar\partial\varphi$.

The estimate also makes sense for non-smooth φ: instead of $|\alpha|^2_{i\partial\bar\partial\varphi}$ one has to take any $H \in L^\infty_{loc}(\Omega)$ with

$$i\bar\alpha \wedge \alpha \le H\, i\partial\bar\partial\varphi$$

(B., 2005).

Theorem. Ω - pseudoconvex in $\mathbb{C}^n$, φ - psh in Ω
$\alpha \in L^2_{loc,(0,1)}(\Omega)$, $\bar\partial\alpha = 0$
$\psi \in W^{1,2}_{loc}(\Omega)$ locally bounded from above, s.th.

$$|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \begin{cases} \leq 1 & \text{in } \Omega \\ \leq \delta < 1 & \text{on } \operatorname{supp}\alpha. \end{cases}$$

Then there exists $u \in L^2_{loc}(\Omega)$ with $\bar\partial u = \alpha$ and

$$\int_\Omega |u|^2(1 - |\bar\partial\psi|^2_{i\partial\bar\partial\varphi})e^{2\psi-\varphi}d\lambda \leq \frac{1+\sqrt{\delta}}{1-\sqrt{\delta}} \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi}e^{2\psi-\varphi}d\lambda.$$

Sketch of proof (ideas going back to Berndtsson and B.-Y. Chen). By approximation we may assume that φ is smooth up to the boundary and strongly psh, and ψ is bounded.
u - minimal solution to $\bar\partial u = \alpha$ in $L^2(\Omega, e^{\psi-\varphi})$
$\Rightarrow u \perp \ker\bar\partial$ in $L^2(\Omega, e^{\psi-\varphi})$
$\Rightarrow v := ue^{\psi} \perp \ker\bar\partial$ in $L^2(\Omega, e^{-\varphi})$
$\Rightarrow v$ - minimal solution to $\bar\partial v = \beta := e^{\psi}(\alpha + u\bar\partial\psi)$ in $L^2(\Omega, e^{-\varphi})$
By Hörmander's estimate

$$\int_\Omega |u|^2 e^{2\psi-\varphi}d\lambda = \int_\Omega |v|^2 e^{-\varphi}d\lambda \leq \int_\Omega |\beta|^2_{i\partial\bar\partial\varphi}e^{-\varphi}d\lambda = \ldots$$

IV. Main Result

Theorem. Ω - pseudoconvex in $\mathbb{C}^{n-1} \times D$, where $0 \in D \subset \mathbb{C}$,
φ - psh in Ω, f - holomorphic in $\Omega' := \Omega \cap \{z_n = 0\}$
Then there exists a holomorphic extension F of f to Ω such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

Sketch of proof. By approximation may assume that Ω is bounded, smooth, strongly pseudoconvex, φ is smooth up to the boundary, and f is holomorphic in a neighborhood of $\overline{\Omega'}$.
$\varepsilon > 0$

$$\alpha := \bar\partial\big(f(z')\chi(-2\log|z_n|)\big),$$

where $\chi(t) = 0$ for $t \leq -2\log\varepsilon$ and $\chi(\infty) = 1$.
$G := G_D(\cdot, 0)$

$$\widetilde{\varphi} := \varphi + 2G + \eta(-2G), \quad \psi := \gamma(-2G).$$

$$F := f(z')\chi(-2\log|z_n|) - u,$$

where u is a solution of $\bar\partial u = \alpha$ given by the previous thm.

V. ODE Problem

Find $g \in C^{0,1}(\mathbb{R}_+)$, $h \in C^{1,1}(\mathbb{R}_+)$ such that

$$\lim_{t\to\infty}(g(t) + \log t) = \lim_{t\to\infty}(h(t) + \log t) = 0$$

and

$$\left(1 - \frac{(g')^2}{h''}\right) e^{2g-h+t} \geq 1.$$

$$h(t) := -\log(t + e^{-t} - 1)$$

$$g(t) := -\log(t + e^{-t} - 1) + \log(1 - e^{-t}).$$

VI. Remaining Open Problem

M - hyperbolic Riemann surface, i.e. it admits a bounded nonconstant subharmonic function. Then $c_M|dz|$ is an invariant metric on M (Suita metric).

Theorem. $Curv_{c_M|dz|} \leq -1$

Conjecture. "$<$" $\Leftrightarrow M \simeq \Delta \setminus F$ (Δ - unit disk, F - polar)