EXTREMAL FUNCTIONS AND EQUILIBRIUM MEASURES
FOR BOREL SETS

by
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Abstract. We extend some results on extremal functions and equilibri-um measures from compact to Borel subsets of $\mathbb{C}^N$.

0. Introduction.

Let $E$ be a bounded Borel set with closure $\overline{E}$ contained in a bounded, hyper-convex domain $\Omega$ in $\mathbb{C}^N$ (i.e., there exists a continuous, negative plurisubharmonic (psh) exhaustion function for $\Omega$). We let

$$u_E(z) := \sup\{u(z) : u \text{ psh in } \Omega, \ u \leq 0, \ u \leq -1 \text{ on } E\}$$

and call $u_E^*(z) := \limsup_{\zeta \to z} u_E(\zeta)$ the relative extremal function of $E$ (relative to $\Omega$). Similarly, letting

$$V_E(z) := \sup\{u(z) : u \in L, \ u \leq 0 \text{ on } E\}$$

where

$$L := \{u \text{ psh in } \mathbb{C}^N : u(z) - \log |z| = 0(1), \ |z| \to \infty\},$$

we call $V_E^*(z) := \limsup_{\zeta \to z} V_E(\zeta)$ the global extremal function of $E$. It is well-known that $u_E^* \equiv 0 \iff V_E^* \equiv +\infty \iff E$ is pluripolar; i.e., there exists $u$ psh in $\mathbb{C}^N$ with $E \subset \{z \in \mathbb{C}^N : u(z) = -\infty\}$. If $E$ is not pluripolar, then, using the complex Monge-Ampere operator $(dd^c(\cdot))^N$ for locally bounded psh functions, we can define the relative and global equilibrium measures $(dd^c u_E^*)^N$ and $(dd^c V_E^*)^N$ for $E$. It is known (cf., [BT1] or [K]) that these measures are supported in $\overline{E}$ and, in the case where $E$ is compact and the polynomially convex hull $\hat{E}$ of $E$ is contained in $\Omega$, $(dd^c u_E^*)^N$ and $(dd^c V_E^*)^N$ are mutually absolutely continuous [L]. Moreover, one can define a nonnegative function $C(E)$ on the Borel subsets $E$ of $\Omega$ via

$$C(E) := \sup\{\int_E (dd^c u)^N : u \text{ psh on } \Omega, \ 0 \leq u \leq 1\}.$$

For Borel sets we have (Proposition 4.7.2 [K])

$$C(E) = \int_{\Omega} (dd^c u_E^*)^N.\label{1}$$

In fact, from Proposition 10.1 [BT1] it follows that

$$C(E) = \int_{\Omega} -u_E^*(dd^c u_E^*)^N.\tag{0.1}$$

The purpose of this note is to give more precise information on the behavior of the extremal functions and extremal measures for Borel sets. First we prove the following equivalences.

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Theorem 1. Let $E$ and $F$ be Borel sets with $E \subset \Omega$ and $F \subset E$. The following statements are equivalent.
1. $C(E) = C(F)$
2. $u_E^* = u_F^*$
3. $V_E^* = V_F^*$.

Next we generalize the mutual absolute continuity of the relative and global equilibrium measures.

Theorem 2. Let $E$ be a nonpluripolar Borel set with $\tilde{E} \subset \Omega$ where $\Omega$ is hyperconvex. Then

$$(\sup_{\partial \Omega} V_E)^{-N} (dd^c V_E^*)^N \leq (dd^c u_E^*)^N \leq (\inf_{\partial \Omega} V_E)^{-N} (dd^c V_E^*)^N.$$  \hfill (0.2)

Note that we have $0 < \inf_{\partial \Omega} V_E \leq \sup_{\partial \Omega} V_E < \infty$ (cf., Proposition 5.3.3 [K]).

Theorem 1 was stated in [BT2] for $E$ compact and also proved in this case by Tom Bloom and is used in [B]. An alternate proof for the planar case ($N = 1$) can be found in [ST], pp. 226-227. The mutual absolute continuity of $(dd^c u_E^*)^N$ and $(dd^c V_E^*)^N$ for $E$ compact was proved in [L].

1. Proof of Theorem 1.

Note that Theorem 1 is trivial if $E$ is pluripolar; thus we assume for the remainder of the discussion that $E$ is nonpluripolar.

Lemma 1.1. Let $E$ be a nonpluripolar Borel set with $\tilde{E} \subset \Omega$. Define

$$E' := \{ z \in \Omega : u_E^*(z) = -1 \}.$$

Then
1. $E' = \{ z \in \Omega : V_E^*(z) = 0 \}$;
2. $u_E^* = u_{E'}^*$ and $V_E^* = V_{E'}^*$.

Proof. (1) follows from Proposition 5.3.3 [K]. For (2), we prove $u_E^* = u_{E'}^*$; the proof for $V_E^* = V_{E'}^*$ is similar. First of all, from the definition of $E'$, we have $u_E^* \leq u_{E'}^* \leq u_{E'}^*$. Since $u_E^* = -1$ on $E$ except perhaps a pluripolar set (cf., Theorem 4.7.6 [K]), we also have that $E \subset E' \cup A$ where $A$ is pluripolar. By Proposition 5.2.5 [K], $u_{E'}^* = u_{E'\cup A}^* \leq u_E^*$ and equality holds. \hfill \clubsuit

Proof of Theorem 1. 1. implies 2: This argument was shown to us by Urban Cegrell. Suppose $C(E) = C(F)$. Since $F \subset E$, $u_E^* \leq u_F^*$. Using this inequality, (0.1), and Stokes’ theorem (recall that $\tilde{E} \subset \Omega$ so that $u_E^*, u_F^* = 0$ on $\partial \Omega$ (cf. [K], Proposition 4.5.2)), we obtain

$$C(E) = \int_{\tilde{E}} -u_E^* (dd^c u_E^*)^N \geq \int_{\Omega} -u_F^* (dd^c u_F^*)^N$$
$$= \int_{\Omega} -u_F^* dd^c u_F^* \wedge (dd^c u_E^*)^{N-1}$$
$$\geq \int_{\Omega} -u_F^* dd^c u_F^* \wedge (dd^c u_E^*)^{N-1}.$$
\[
\cdots \geq \int_{\Omega} -u^*_F(dd^cu^*_F)^N = C(F).
\]
Thus equality holds throughout; in particular, from the first line we have
\[
\int_{\Omega} (u^*_E - u^*_F)(dd^cu^*_E)^N = 0; \text{ i.e.,}
\]
\[
\int_{\{u^*_E < u^*_F\}} (dd^cu^*_E)^N = 0.
\]
By the comparison principle (Corollary 3.7.5 [K]), \(u^*_E \geq u^*_F\) and equality holds.

2. \(\iff\): Let \(E' := \{z \in \Omega : u^*_E(z) = -1\}\) and \(F' := \{z \in \Omega : u^*_F(z) = -1\}\).

From (1) and (2) of Lemma 1.1 it suffices to show that
\[u^*_E = u^*_F \iff E' = F'\]
(the proof that \(V^*_E = V^*_F \iff E' = F'\) is similar). The implication \(u^*_E = u^*_F\)
implies \(E' = F'\) is obvious. For the reverse implication, if \(E' = F'\), then \(u^*_E = -1\)
on \(F' = E'\) so that \(u^*_E \leq u^*_F \leq u^*_E\) = \(u^*_E\) (the last equality is from (2) of Lemma
1.1). The reverse inequality follows since \(F \subset E\).

2. Proof of Theorem 2.

In this section, we prove Theorem 2, the mutual absolute continuity of the equi-
librium measures \((dd^cu^*_E)^N\) and \((dd^V^*_E)^N\) when \(E\) is a nonpluripolar Borel set.
The main tool will be the following result.

Lemma 2.1. Let \(E\) be a compact subset of a bounded domain \(\Omega\) in \(\mathbb{C}^N\). Let \(u_1, u_2\)
be nonnegative continuous functions on \(\overline{\Omega}\) which are psh on \(\Omega\). If
(1) \(u_1 = u_2 = 0\) on \(E\);
(2) \(u_1 \geq u_2\) on \(\Omega\);
(3) \((dd^cu_1)^N = (dd^cu_2)^N = 0\) on \(\Omega \setminus E\);
(4) \(u_2 > 0\) on \(\partial \Omega\),
then \((dd^cu_1)^N \geq (dd^cu_2)^N\); i.e., for all \(\phi \in C^\infty_0(\Omega)\) with \(\phi \geq 0\),
\[
\int_{\Omega} \phi(dd^cu_1)^N \geq \int_{\Omega} \phi(dd^cu_2)^N.
\]

Proof. This lemma follows easily from Theorem 5.6.5 [K] (see also [L]). For let \(\omega\)
be a domain containing \(E\) such that \(\overline{\omega} \subset \Omega\) and \(u_2 > 0\) on \(\partial \omega\). Take any positive \(t\)
with \(t < 1\). Then we have \(u_1 \geq tu_2 + \eta\) on \(\partial \omega\) for some \(\eta > 0\). By Theorem 5.6.5
[K], \((dd^cu_1)^N \geq (dd^c(tu_2))^N\) and the lemma follows.

We shall also need two simple lemmas.

Lemma 2.2. Let \(E\) and \(\Omega\) be as in Theorem 2. Then
\[
\sup_{\partial \Omega} V_E = \sup_{\overline{\Omega}} V^*_E.
\]

Proof. The inequality \(\sup_{\partial \Omega} V_E \leq \sup_{\overline{\Omega}} V^*_E\) is obvious. To show the reverse inequality take any \(u \in L\) with \(u \leq 0\) on \(E\). Then \(u \leq \sup_{\partial \Omega} V_E\) on \(\overline{\Omega}\); hence
\(V_E \leq \sup_{\partial \Omega} V_E\) on \(\overline{\Omega}\) so that \(\sup_{\overline{\Omega}} V^*_E \leq \sup_{\partial \Omega} V_E\).
Lemma 2.3. Let \( \{f_j\} \) be a sequence of lower (resp. upper) semicontinuous functions defined on a compact set \( K \) which increase (resp. decrease) to a bounded function \( f \). Then

\[
\lim_{j \to \infty} \left( \inf_{K} f_j \right) = \inf_{K} f \quad \text{(resp. } \lim_{j \to \infty} \left( \sup_{K} f_j \right) = \sup_{K} f) \).
\]

Proof. We have \( \inf_K f_j \uparrow a \leq \inf_K f \). To prove the reverse inequality, assume that \( a < b < \inf_K f \) for some \( b \). From the lower semicontinuity of the \( \{f_j\} \) it follows that the nonempty sets \( \{f_j \leq b\} \) are compact. However, these sets decrease to the empty set since \( \inf_K f > b \); this is a contradiction. The corresponding statement for a decreasing sequence of upper semicontinuous functions \( \{f_j\} \) follows from the previous argument applied to the functions \( \{-f_j\} \).

Proof of Theorem 2. First assume that \( E \) is compact and L-regular; i.e., \( V_E = V_E^* \). Then \( V_E \) and \( u_E \) are continuous in \( \Omega \) and (0.2) follows from Lemma 2.1, since

\[
V_E / \sup_{\partial \Omega} V_E \leq u_E + 1 \leq V_E / \inf_{\partial \Omega} V_E
\]

(cf., Proposition 5.3.3 [K]).

Now suppose that \( E \) is compact but not necessarily L-regular. For \( j = 1, 2, \ldots \) define \( E_j := \{ z \in E : \text{dist}(z, E) \leq 1/j \} \). Then for \( j \) sufficiently large \( E_j \subset \Omega \) and \( E_j \) is L-regular (Corollary 5.1.5 [K]). Furthermore, \( E_j \uparrow E \), \( u_{E_j} \uparrow u_E \), and \( V_{E_j} \uparrow V_E \) as \( j \uparrow \infty \). Moreover, \( \sup_{\partial \Omega} V_{E_j} \leq \sup_{\partial \Omega} V_E \) and, by Lemma 2.3, \( \lim_{j \to \infty} (\inf_{\partial \Omega} V_{E_j}) = \inf_{\partial \Omega} V_E \). From the previous case and the continuity of the Monge-Ampère operator under monotone increasing limits (cf., Theorem 3.6.1 [K] or Proposition 5.2 [BT1]), we get (0.2) for general nonpluripolar compact sets.

Finally, let \( E \) be an arbitrary nonpluripolar Borel set. Then from Corollary 8.5 [BT1] it follows that there exist compact sets \( E_j, j = 1, 2, \ldots \) and an \( F \) set \( F \) such that \( E_j \uparrow F \subset E \), \( u_{E_j} \uparrow u_F \), \( u_F = u_E^* \), and \( V_{E_j}^* \downarrow V_F^* = V_E^* \). Then \( \inf_{\partial \Omega} V_{E_j} \geq \inf_{\partial \Omega} V_E \) and by Lemmas 2.2 and 2.3,

\[
\lim_{j \to \infty} \left( \sup_{\partial \Omega} V_{E_j} \right) = \lim_{j \to \infty} \left( \sup_{\partial \Omega} V_{E_j}^* \right) = \sup_{\partial \Omega} V_E^* = \sup_{\partial \Omega} V_E.
\]

Using the continuity of the Monge-Ampère operator under monotone decreasing limits (cf., Theorem 3.4.3 [K] or Theorem 2.1 [BT1]), we conclude the proof of the theorem.

References


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