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Estimates for the Complex Monge-Ampère Operator

by

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Summary. An inequality concerning complex Monge-Ampère operator is proven. The L^n-L^1-stability of the Dirichlet problem in $\mathbb{C}^n$ and a result concerning maximal plurisubharmonic functions is deduced from it.

1. Introduction. Throughout this note we use the standard notation $d = \partial + \overline{\partial}$, $d^c = i(\overline{\partial} - \partial)$. Then $dd^c = 2i\partial\overline{\partial}$ and

$$(dd^c)^n = n!4^n \det\left(\frac{\partial^2}{\partial z_j \partial \overline{z}_k}\right) d\lambda$$

for smooth functions; $d\lambda$ stands for the $2n$-dimensional volume form. Our main result is the following inequality

$$(1)\qquad \int_\Omega (h-u)^n dd^c v_1 \wedge \ldots \wedge dd^c v_n \leqslant n! \|v_1\|_\Omega \ldots \|v_{n-1}\|_\Omega \int_\Omega |v_n| (dd^c u)^n,$$

where $u, h, v_1, \ldots, v_n$ are plurisubharmonic (for short psh), locally bounded functions in a bounded domain Ω in $\mathbb{C}^n$ such that $u \leqslant h$ in Ω and $u = h$ on $\partial\Omega$ and $v_1, \ldots, v_n$ are negative (see Theorem 2.1 below for details).

As an application of the inequality (1) we obtain two results. First is the L^n-L^1-stability of the Dirichlet problem in $\mathbb{C}^n$ (Theorem 3.1). Cegrell and Persson [5] obtained the L^∞-L^2-stability. The second application of (1) is a generalization of a theorem of Sadullaev concerning maximal psh functions (see Theorem 4.4). From (1) one can also deduce a generalization of Chern-Levine-Nirenberg inequalities [6] due to Demailly [7] (see Corollary 2.2 below).

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2. Proof of the main result.

THEOREM 2.1. *Let Ω be a bounded domain in $\mathbb{C}^n$. Let $u, h, v_1, \ldots, v_n \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$ be such that $u \leqslant h$, $\lim_{z\to\partial\Omega}(h(z) - u(z)) = 0$ and $v_j \leqslant 0$ $(j = 1, \ldots, n)$. Then* (1) *holds.*

Proof. Taking $h_\varepsilon := \max\{u, h - \varepsilon\}$ we see that the left-hand side of (1) with h replaced by h_ε converges. We can therefore assume that $h = u$ in a neighbourhood of $\partial\Omega$.

Assume we have proven the theorem for smooth u and h. Now for arbitrary u and h let u_ε and h_ε denote their smooth regularizations, so that $u_\varepsilon \downarrow u$, $h_\varepsilon \downarrow h$ as $\varepsilon \downarrow 0$. Then $u_\varepsilon \leqslant h_\varepsilon$ and, if we shrink Ω a little, $u_\varepsilon = h_\varepsilon$ in a neighbourhood of $\partial\Omega$. So we have

$$(2) \quad \int_\Omega (h_\varepsilon - u_\varepsilon)^n dd^c v_1 \wedge \ldots \wedge dd^c v_n \leqslant n! \|v_1\|_\Omega \ldots \|v_{n-1}\|_\Omega \int_\Omega |v_n| (dd^c u_\varepsilon)^n.$$

By Lebesgue's Dominated Convergence Theorem, the left-hand side of (2) tends to

$$\int_\Omega (h - u)^n dd^c v_1 \wedge \ldots \wedge dd^c v_n.$$

By the convergence theorem from [3], the measures $v_n(dd^c u_\varepsilon)^n$ converge weakly to $v_n(dd^c u)^n$ thus, having shrunk Ω already, the right-hand side of (2) tends to

$$n! \|v_1\|_\Omega \ldots \|v_{n-1}\|_\Omega \int_\Omega |v_n| (dd^c u)^n.$$

Now it suffices to prove Theorem 2.1 when u and h are smooth and they coincide in a neighbourhood of $\partial\Omega$. We may also assume $\|v_j\|_\Omega = 1$. First, we want to show that for $p = 2, \ldots, n$ we have

$$(3) \quad \int_\Omega (h - u)^p (dd^c u)^{n-p} \wedge dd^c v_{n-p+1} \wedge \ldots \wedge dd^c v_n$$
$$\leqslant p \int_\Omega (h - u)^{p-1} (dd^c u)^{n-p+1} \wedge dd^c v_{n-p+2} \wedge \ldots \wedge dd^c v_n.$$

By Stokes's Theorem, the left-hand side of (3) is equal to

$$\int_\Omega v_{n-p+1} (dd^c u)^{n-p} \wedge dd^c (h - u)^p \wedge dd^c v_{n-p+2} \wedge \ldots \wedge dd^c v_n,$$

while the right-hand side of (3) is greater than or equal to

$$p \int_\Omega (-v_{n-p+1})(h - u)^{p-1} (dd^c u)^{n-p+1} \wedge dd^c v_{n-p+2} \wedge \ldots \wedge dd^c v_n,$$

because $-v_{n-p+1} = |v_{n-p+1}| \leqslant 1$. To show (3) it is therefore enough to prove that for $p = 2, \ldots, n$

$$(4) \quad -dd^c(h - u)^p \leqslant p(h - u)^{p-1} dd^c u,$$

that is the difference between the right- and the left-hand side of (4) is a positive (1,1)-form. We have

$$\begin{aligned} dd^c(h - u)^p &= d(p(h - u)^{p-1} d^c(h - u)) \\ &= p(p - 1)(h - u)^{p-2} d(h - u) \wedge d^c(h - u) \\ &\quad + p(h - u)^{p-1} dd^c(h - u) \\ &\geqslant p(p - 1)(h - u)^{p-2} d(h - u) \wedge d^c(h - u) - p(h - u)^{p-1} dd^c u \end{aligned}$$

and $d(h - u) \wedge d^c(h - u) = i\partial(h - u) \wedge \overline{\partial}(h - u) \geqslant 0$, thus (4) follows and we have (3). Now, by (3) and Stokes's Theorem

$$\begin{aligned} \int_\Omega (h - u)^n dd^c v_1 \wedge \ldots \wedge dd^c v_n &\leqslant n! \int_\Omega (h - u)(dd^c u)^{n-1} \wedge dd^c v_n \\ &= n! \int_\Omega (-v_n)(dd^c u)^{n-1} \wedge dd^c (u - h) \\ &\leqslant n! \int_\Omega |v_n| (dd^c u)^n \end{aligned}$$

and the proof is complete.

COROLLARY 2.2. *Let $\Omega \Subset \mathbb{C}^n$ and let $u, v_1, \ldots, v_n \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$ be negative. Assume that $\lim_{z\to\partial\Omega} u(z) = 0$. Then*

$$\int_\Omega |u|^n dd^c v_1 \wedge \ldots \wedge dd^c v_n \leqslant n! \|v_1\|_\Omega \ldots \|v_{n-1}\|_\Omega \int_\Omega |v_n| (dd^c u)^n.$$

(compare with [7, Th. 2.2]).

COROLLARY 2.3. *Let Ω be a bounded domain in $\mathbb{C}^n$. Take $u, h \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$ such that $u \leqslant h$ and $\lim_{z\to\partial\Omega}(h(z) - u(z)) = 0$. Then*

$$\|h - u\|_{L^n(\Omega)} \leqslant R^2/4 \Big(\int_\Omega (dd^c u)^n \Big)^{1/n},$$

where

$$R = \min\{r > 0 : \Omega \subset B(z_0, r) \quad \textit{for some } z_0 \in \mathbb{C}^n\}.$$

In particular $R \leqslant \operatorname{diam}\Omega$, and when Ω is a ball then $R = \operatorname{diam}\Omega/2$.

Proof. Without loss of generality we may assume that $\Omega \subset B(0, R)$. It is enough to apply Theorem 2.1 with

$$v_1(z) = \ldots = v_n(z) = |z|^2 - R^2.$$

Remark. Certainly if Ω fulfils the assumptions of Corollary 2.2 with $u \not\equiv 0$, then it must be hyperconvex, in particular pseudoconvex. Notice that this is not the case in Corollary 2.3. In fact in the proof of Theorem 4.4 below we shall use Corollary 2.3 for domains which are not necessarily pseudoconvex.

To understand the meaning of Corollary 2.3 consider the following situation: take $u \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$ and $G \Subset \Omega$. Then by Proposition 4.1 below there exists a unique solution to the following Dirichlet problem

$$\begin{cases} h \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega) \\ h \text{ maximal in } G \\ h = u \text{ in } \Omega \setminus \overline{G}. \end{cases}$$

Then Theorem 2.3 says, roughly speaking, that if the measure $(dd^c u)^n$ is small in a neighbourhood of $\overline{G}$ then the functions h and u do not differ much in the L^n-norm.

3. The L^n-L^1-stability of the Dirichlet problem in $\mathbb{C}^n$. Let Ω be strictly pseudoconvex in $\mathbb{C}^n$, $\varphi \in C(\partial\Omega)$ and $f \in C(\overline{\Omega})$, $f \geqslant 0$. Then the following Dirichlet problem

$$\begin{cases} u \in C(\overline{\Omega}) \cap \mathrm{PSH}(\Omega) \\ (dd^c u)^n = f d\lambda \\ u|_{\partial\Omega} = \varphi \end{cases}$$

has a unique solution $U(\varphi, f)$ (cf.[2]). Cegrell and Persson [5], using a connection between real and complex Monge-Ampère operators (due to the idea of Cheng and Yau discussed in [1]), proved that

$$\|U(\varphi_1, f_1) - U(\varphi_2, f_2)\|_\Omega \leqslant \|\varphi_1 - \varphi_2\|_{\partial\Omega} + C\|f_1 - f_2\|^{1/n}_{L^2(\Omega)}.$$

We prove

THEOREM 3.1. *With the above notations we have*

$$\|U(\varphi_1, f_1) - U(\varphi_2, f_2)\|_{L^n(\Omega)} \leqslant \lambda(\Omega)\|\varphi_1 - \varphi_2\|_{\partial\Omega} + C\|f_1 - f_2\|^{1/n}_{L^1(\Omega)} \tag{5}$$

with $C = R^2/4$, as in Corollary 2.3.

Proof. This is in fact an easy application of Corollary 2.3. We have

$$\|U(0, |f_1 - f_2|)\|_{L^n(\Omega)} \leqslant C\|f_1 - f_2\|^{1/n}_{L^1(\Omega)} \tag{6}$$

and we can use an idea from [5]: from the superadditivity of $(dd^c)^n$ and comparison principle (cf. [3]) it follows that

$$\begin{aligned} |U(\varphi_1, f_1) - U(\varphi_2, f_2)| &\leqslant -U(-|\varphi_1 - \varphi_2|, |f_1 - f_2|) \\ &\leqslant \|\varphi_1 - \varphi_2\|_{\partial\Omega} - U(0, |f_1 - f_2|) \end{aligned} \tag{7}$$

(cf. [5, Cor. 1]). Combining (6) and (7) we get (5).

Remark. If we have the L^{p_0}-L^{q_0} stability then, by Hölder's inequality, we have also the L^p-L^q stability for $1 \leqslant p \leqslant p_0$ and $q_0 \leqslant q \leqslant \infty$. By the above results we have the L^p-L^q stability in $\mathbb{C}^n$ if

$$(p, q) \in \big([1, \infty] \times [2, \infty]\big) \cup \big([1, n] \times [1, \infty]\big).$$

Of course we have not the L^∞-L^1 stability (take $u(z) = \log|z|$ and consider its regularizations). The author does not know whether there is stability for other pairs (p, q).

4. Maximal plurisubharmonic functions. If Ω is a domain in $\mathbb{C}^n$ then we say that a plurisubharmonic function u in Ω is *maximal* if for every open $G \Subset \Omega$ and $v \in \mathrm{PSH}(G)$ such that $v^* \leqslant u$ on ∂G (v^* denotes the upper regularization of v; it is then defined on $\overline{G}$) it follows that $v \leqslant u$ in G. The connection between psh and maximal psh functions is the same as between subharmonic and harmonic functions: psh functions are sub-maximal. If $n = 1$ then maximal functions are harmonic. The notion of maximality was first introduced by Sadullaev [9]; for the basic properties see [8].

The next proposition is essentially known.

PROPOSITION 4.1. *For $u \in \mathrm{PSH}(\Omega)$ and open $G \Subset \Omega$ let*

$$h = h_{u,G} := \sup\{v \in \mathrm{PSH}(\Omega) : v = u \text{ in } \Omega \setminus \overline{G}\}. \tag{8}$$

Then the following hold

(a) $h \in \mathrm{PSH}(\Omega)$,
(b) *h is maximal in G*,
(c) *$h = u$ in $\Omega \setminus \overline{G}$*,
(d) *if $u_j \downarrow u$ and $\overline{G}_j \downarrow \overline{G}$ then $h_{u_j,G_j} \downarrow h_{u,G}$*,
(e) *if $u \in L^\infty_{\mathrm{loc}}(\Omega)$ then $h \in L^\infty_{\mathrm{loc}}(\Omega)$*,
(f) *if $u|_{\overline{G}}$ is continuous in a neighbourhood of ∂G and ∂G is smooth then $h \in C(\overline{G})$; in particular if $u \in C(\Omega)$ and ∂G is smooth then $h \in C(\Omega)$.*

Proof. We sketch the proof for the convenience of the reader. To show (a) it is enough to see that $h^* \leqslant h$ thus $h = h^* \in \mathrm{PSH}(\Omega)$. Now properties (b)–(e) are obvious. It remains to prove (f). We have

$$h|_G = \sup\{v \in \mathrm{PSH}(G) : v^* \leqslant u \text{ on } \partial G\} \tag{9}$$

for if v is as in (9) then

$$\widetilde{v} := \begin{cases} \max\{u, v\} & \text{in } G \\ u & \text{in } \Omega \setminus G \end{cases}$$

is psh in Ω. It follows from the classical potential theory that there is $\widetilde{h} \in C(\overline{G})$, harmonic in G such that $\widetilde{h}|_{\partial G} = u|_{\partial G}$. We see that $u \leqslant h \leqslant \widetilde{h}$ in G

thus $h|_G$ attains continuously boundary values of $u|_{\partial G}$. Now (f) follows from (9) and a theorem of Walsh ([10, Lemma 1]; see also the proof of Theorem 3.1.4 in [8] for the arguments from [10]).

Directly from Proposition 4.1 we get

COROLLARY 4.2. *The set of points where a maximal psh function is not continuous is either empty or it is not relatively compact.*

COROLLARY 4.3. *Every maximal psh function can be approximated by continuous maximal functions. More precisely: If u is maximal in Ω and $G \Subset \Omega$ then one can find a sequence $\{u_j\}$ of continuous maximal psh functions in G such that $u_j \downarrow u$ in G.*

Sadullaev proved Corollary 4.3 for pseudoconvex Ω (cf. [9, 17.3]). It is a result of Bedford and Taylor [2, 3] (see also [9]) that if u is psh and locally bounded then it is maximal if and only if $(dd^c u)^n = 0$. For $n \geqslant 2$ maximal psh functions are not necessarily locally bounded; for example functions of the type $\log|F|$, where F is holomorphic (they are maximal because $(dd^c)^n|F|^2 = 0$ for $n \geqslant 2$); also psh functions which do not depend on one variable are maximal.

For maximal psh functions which are not necessarily locally bounded we have the following

THEOREM 4.4. *Let $\{u_j\} \subset \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$ be a sequence decreasing to $u \in \mathrm{PSH}(\Omega)$. Then u is maximal if $(dd^c u_j)^n$ converges weakly to 0.*

As pointed out to the author by S. Kołodziej one can apply an example due to Cegrell [4] to show that the converse of Theorem 1.1 is not true if we do not assume u to be bounded. Namely, the functions

$$u_j(z) = \sum_{k=1}^{n} \max\{\log|z_k|, -j\}$$

are psh and decrease to $u(z) = \log|z_1 \ldots z_n|$, which is maximal for $n \geqslant 2$. However $(dd^c u_j)^n$ tends weakly to $n!(2\pi)^n\delta_0$, where δ_0 is the Dirac measure.

Here we give an alternative proof of Theorem 4.4 quite different from the Sadullaev's one.

Proof of Theorem 4.4. It is enough to take an open $G \Subset \Omega$ with smooth boundary and to check that u is maximal in G. Let $h_j = h_{u_j,G}$ be defined by (8). Then $h_j \downarrow h = h_{u,G}$ and it is enough to show that $h = u$. By Corollary 2.3 the functions $h_j - u_j$ converge to 0 in L^n-norm and thus there is a subsequence converging almost everywhere. Therefore $h = u$ almost everywhere and hence everywhere. The proof of Theorem 1.1 is complete.

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