Remark on the definition of the complex Monge-Ampère operator

Zbigniew Blocki

Dedicated to Vyacheslav P. Zakharyuta on the occasion of his 70th birthday

Abstract. We show that if the function \( \chi : \mathbb{R} \rightarrow \mathbb{R} \) is increasing, convex, and satisfies
\[
\int_{-\infty}^{-1} (-\chi(t))^{n-2}(\chi'(t))^2 \, dt < \infty, \quad n \geq 2,
\]
then for any plurisubharmonic \( u \) the complex Monge-Ampère operator \((dd^c)^n\) is well defined for the plurisubharmonic function \( \chi \circ u \). The condition on \( \chi \) is optimal.

1. Introduction

In [2] and [3] the domain of definition \( D \) for the complex Monge-Ampère operator \((dd^c)^n\) was defined as follows: we say that a plurisubharmonic function \( u \) belongs to \( D \) if there is a regular measure \( \mu \) such that for any sequence \( u_j \) of smooth plurisubharmonic functions decreasing to \( u \) the Monge-Ampère measures \((dd^c u_j)^n\) converge weakly to \( \mu \). (In this definition we consider germs of functions on \( \mathbb{C}^n \), so that the approximating sequence \( u_j \) may be defined on a smaller domain than \( \mu \) is.) It was for example shown in [2], [3] that if \( D \ni u \leq v \in PSH \) then \( v \in D \), and that for \( n = 2 \) we have \( D = PSH \cap W^{1,2}_{loc} \).

In this note we show the following result (we always assume \( n \geq 2 \)):

**Theorem 1.** Assume that \( \chi : \mathbb{R} \rightarrow \mathbb{R} \) is increasing, convex, and such that
\[
\int_{-\infty}^{-1} (-\chi(t))^{n-2}(\chi'(t))^2 \, dt < \infty.
\]

Then for any plurisubharmonic \( u \) we have \( \chi \circ u \in D \).

The assumptions in Theorem 1 are for example satisfied for the function \( \chi(t) = -(-t)^{\alpha} \) (for \( t \leq -1 \)), where \( 0 < \alpha < 1/n \). As an immediate consequence of Theorem

**2000 Mathematics Subject Classification.** 32W20, 32U05.

**Key words and phrases.** Complex Monge-Ampère operator, plurisubharmonic functions.

This paper was written during the author stay at the Institut Mittag-Leffler (Djursholm, Sweden). It was also partially supported by the projects NN2 01367933 and 189/6 PR EU/2007/7 of the Polish Ministry of Science and Higher Education

©2009 American Mathematical Society
we thus obtain the following property of pluripolar sets (compare with Theorem 5.8 in [4]):

**Corollary.** If $E \subset \mathbb{C}^n$ is pluripolar then $E \subset \{ u = -\infty \}$ for some $u \in \mathcal{D}(\mathbb{C}^n)$.

The main tool in the proof will be the following characterization of the class $\mathcal{D}$ (see [3]): for a negative plurisubharmonic function $u$ we have $u \in \mathcal{D}$ if and only if there exists a sequence (or equivalently: for every sequence) $u_j \in PSH \cap C^\infty$ decreasing to $u$ the sequences

\[(2) \quad (-u_j)^{n-2-k} du_j \wedge d^c u_j \wedge (dd^c u_j)^k \wedge \omega^{n-1-k}, \quad k = 0, 1, \ldots, n-2,\]

are locally uniformly weakly bounded (here $\omega := dd^c |z|^2$).

It follows easily from (2) that (1) is an optimal condition: if $\chi(\log |z_1|) \in \mathcal{D}$ then by (2) for $k = 0$ we have

\[
\int_{\{|z| < \epsilon\}} \frac{(-\chi(\log |z|))^{n-2}(\chi'(\log |z|))^2}{|z|^2} d\lambda(z) < \infty,
\]

which is equivalent to

\[
\int_{-\infty}^{\log \epsilon} (-\chi(t))^{n-2}(\chi'(t))^2 dt < \infty.
\]

A result related to Theorem 1 has been proved by Bedford and Taylor (see [1], p. 66-69). They namely showed the following:

**Theorem 2.** Assume that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing and such that

\[
\int_1^\infty \frac{\phi(x)}{x} dx < \infty.
\]

Let $v$ be a plurisubharmonic function such that for some negative plurisubharmonic $u$ we have $-(-u \phi \circ u)^{1/n} \leq v$. Then $v \in \mathcal{D}$.

We will now show how Theorem 1 implies Theorem 2. Set

\[
\gamma(t) := -\frac{1}{2} \int_0^t \sqrt{\frac{\phi(-s)}{-s}} ds, \quad t \leq 0.
\]

Then

\[
\gamma'(t) = \frac{1}{2} \sqrt{\frac{\phi(-t)}{-t}}
\]

and thus $\gamma : \mathbb{R}_- \to \mathbb{R}_-$ is convex and increasing. Moreover,

\[
\frac{d}{dt} \left(-(-t\phi(-t))^{1/2}\right) = \frac{1}{2} \left(-(-t\phi(-t))^{1/2}\right)^{-1/2} \left(\phi(-t) - t\phi'(-t)\right) \leq \gamma'(t).
\]

Therefore $\gamma(t) \leq -(-t\phi(-t))^{1/2}, t \leq 0$. Thus

\[
\chi(t) := -(-\gamma(t))^{2/n} \leq -(-t\phi(-t))^{1/n},
\]

$\chi : \mathbb{R}_- \to \mathbb{R}_-$ is convex and increasing, and

\[
\int_{-\infty}^{-1} (-\chi(t))^{n-2}(\chi'(t))^2 dt = \frac{4}{n^2} \int_{-\infty}^{-1} (\gamma'(t))^2 dt < \infty.
\]

By Theorem 1 we have $\chi \circ u \in \mathcal{D}$ and it is now enough to apply Theorem 1.2 in [3].
The author would like to thank the organizers of the Complex Analysis and Functional Analysis conference held at Sabancı University in Istanbul in September 2007 for the invitation. He is also grateful to Azim Sadullaev for his interest in the problem.

**Proof**

Theorem 1 will be proved by successive application of the following estimates:

**Lemma.** Let \( \gamma : \mathbb{R} \to \mathbb{R}_+ \) be continuous and such that \( \int_{-\infty}^{0} \gamma(t) \, dt < \infty. \) Set

\[
    f(t) := \int_{-\infty}^{t} \gamma(s) \, ds, \quad g(t) := \int_{t}^{0} f(s) \, ds, \quad t < 0,
\]

so that \( f, g \geq 0, \) \( f' = \gamma, \) \( g' = -f. \) Assume that \( K \subset \Omega, \) where \( \Omega \) is a domain in \( \mathbb{C}^n. \) Let \( T, S \) be closed positive currents in \( \Omega \) of bidegree, respectively, \((n-1,n-1)\) and \((n-2,n-2).\) Then for any negative \( u \in \text{PSH} \cap C^\infty(\Omega) \) we have

\[
    \int_K \gamma \circ u \, du \wedge d^c u \wedge T \leq C_1 \int_{\Omega} g \circ u \omega \wedge T, \tag{3}
\]

\[
    \int_K \gamma \circ u \, du \wedge d^c u \wedge dd^c u \wedge S \leq C_2 \int_{\Omega} f \circ u \, du \wedge d^c u \wedge \omega \wedge S, \tag{4}
\]

where \( C_1, C_2 \) are positive constants depending only on \( K \) and \( \Omega. \)

**Proof.** Let \( \varphi \) be a nonnegative test function in \( \Omega \) with \( \varphi = 1 \) on \( K. \) Then

\[
\begin{align*}
    \int_K \gamma \circ u \, du \wedge d^c u \wedge T &\leq \int_{\Omega} \varphi \gamma \circ u \, du \wedge d^c u \wedge T \\
    &\leq \int_{\Omega} \varphi \, d(f \circ u) \wedge d^c u \wedge T \\
    &\leq -\int_{\Omega} \varphi f \circ u \, dd^c u \wedge T - \int_{\Omega} f \circ u \, d\varphi \wedge d^c u \wedge T \\
    &\leq -\int_{\Omega} f \circ u \, d\varphi \wedge d^c u \wedge T \\
    &\leq \int_{\Omega} d\varphi \wedge d^c (g \circ u) \wedge T \\
    &\leq -\int_{\Omega} g \circ u \, dd^c \varphi \wedge T \\
    &\leq C_1 \int_{\Omega} g \circ u \omega \wedge T.
\end{align*}
\]

To show (4) we start the same way:

\[
\begin{align*}
    \int_K \gamma \circ u \, du \wedge d^c u \wedge dd^c u \wedge S &\leq -\int_{\Omega} g \circ u \, dd^c \varphi \wedge dd^c u \wedge S \\
    &\leq -\int_{\Omega} f \circ u \, du \wedge d^c u \wedge dd^c \varphi \wedge S \\
    &\leq C_2 \int_{\Omega} f \circ u \, du \wedge d^c u \wedge \omega \wedge S. \qed
\end{align*}
\]
Proof of Theorem 1. Without loss of generality we may assume that $u \leq -1$ and $\chi(0) = 0$ (because subtracting a constant from $\chi$ does not change (1)). For $k = 0, 1, \ldots, n - 2$ we set $\gamma_k := (-\chi)^{n-2-k}(\chi')^{k+2}$. Our goal is to show that for $K \Subset \Omega \subset \mathbb{C}^n$ and $u \in PSH \cap C^\infty(\Omega)$, $u \leq -1$, the following estimate holds

$$\int_K \gamma_k \circ u \, du \wedge (dd^c u)^k \wedge \omega^{n-k-1} \leq C \int_{-\infty}^{-1} \gamma_0(t) \, dt \, ||u||_{L^1(\Omega)},$$

where $C$ is a positive constant depending only on $K$ an $\Omega$. In view of (2) this will finish the proof.

By $F$ denote the class of those $\gamma$ that satisfy the assumptions of Lemma, that is $\gamma : \mathbb{R} \rightarrow \mathbb{R}^+$ is continuous and $\int_{-\infty}^{1} - \infty < \gamma(t) \, dt < \infty$. For $\gamma \in F$ we also define $$(F^{\gamma})_t := \int_{-\infty}^{t} \gamma(s) \, ds, \quad t \in \mathbb{R},$$

and $F^l \gamma := F \cdots F \gamma$. Note that since $\chi'(s) \leq \chi(s)/s$ for $s < 0$, we have $\gamma_k \in F$ by (1). We claim that $F^{\gamma_k} \in F$ for $k \geq 1$. For $a < 0$ by the Fubini theorem we have

$$(F^2 \gamma_k)(a) = \int_a^a \int_{-\infty}^{t} \gamma_k(s) \, ds \, dt = \int_{-\infty}^{a} \int_s^a \gamma_k(s) \, dt \, ds \leq - \int_{-\infty}^{a} s \gamma_k(s) \, ds.$$

Hence it follows that for $k = 1, \ldots, n - 2$

$$F^2 \gamma_k \leq F^{\gamma_k} \quad \text{on} \quad \mathbb{R}^-.$$

This implies that $F^l \gamma_k \in F$, $l = 1, \ldots, k + 1$, and

$$F^{k+1} \gamma_k \leq (F^{\gamma_0})(-1) = \int_{-\infty}^{-1} \gamma_0(t) \, dt \quad \text{on} \quad (-\infty, -1].$$

Using (4) $k$ times we will get

$$\int_K \gamma_k \circ u \, du \wedge (dd^c u)^k \wedge \omega^{n-k-1} \leq C(K, \Omega') \int_{\Omega'} (F^k \gamma_k) \circ u \, du \wedge d^c u \wedge \omega^{n-1},$$

where $K \Subset \Omega' \Subset \Omega$. Now set

$$g(t) := \int_{t}^{0} (F^{k+1} \gamma_k)(s) \, ds, \quad t < 0.$$

Then by (3)

$$\int_K \gamma_k \circ u \, du \wedge d^c u \wedge (dd^c u)^k \wedge \omega^{n-k-1} \leq C(K, \Omega) \int_{\Omega} g \circ u \omega^n,$$

and by (6)

$$g(t) \leq |t| \int_{-\infty}^{-1} \gamma_0(s) \, ds, \quad t < 0.$$

We thus obtain (5). □
DEFINITION OF THE COMPLEX MONGE-AMPERE OPERATOR

References


Jagiellonian University, Institute of Mathematics, Reymonta 4, 30-059 Kraków, Poland

E-mail address: Zbigniew.Blocki@im.uj.edu.pl