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Remark on the definition of the complex Monge-Ampère operator

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Dedicated to Vyacheslav P. Zakharyuta on the occasion of his 70th birthday

Abstract. We show that if the function $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ is increasing, convex, and satisfies $\int_{-\infty}^{-1}(-\chi(t))^{n-2}(\chi'(t))^2 dt < \infty$, $n \geq 2$, then for any plurisubharmonic u the complex Monge-Ampère operator $(dd^c)^n$ is well defined for the plurisubharmonic function $\chi \circ u$. The condition on χ is optimal.

1. Introduction

In [2] and [3] the domain of definition $\mathcal{D}$ for the complex Monge-Ampère operator $(dd^c)^n$ was defined as follows: we say that a plurisubharmonic function u belongs to $\mathcal{D}$ if there is a regular measure μ such that for any sequence u_j of smooth plurisubharmonic functions decreasing to u the Monge-Ampère measures $(dd^c u_j)^n$ converge weakly to μ. (In this definition we consider germs of functions on $\mathbb{C}^n$, so that the approximating sequence u_j may be defined on a smaller domain than μ is.) It was for example shown in [2], [3] that if $\mathcal{D} \ni u \leq v \in PSH$ then $v \in \mathcal{D}$, and that for $n = 2$ we have $\mathcal{D} = PSH \cap W^{1,2}_{loc}$.

In this note we show the following result (we always assume $n \geq 2$):

Theorem 1. *Assume that $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ is increasing, convex, and such that*

$$\int_{-\infty}^{-1}(-\chi(t))^{n-2}(\chi'(t))^2 dt < \infty. \tag{1}$$

Then for any plurisubharmonic u we have $\chi \circ u \in \mathcal{D}$.

The assumptions in Theorem 1 are for example satisfied for the function $\chi(t) = -(-t)^\alpha$ (for $t \leq -1$), where $0 < \alpha < 1/n$. As an immediate consequence of Theorem

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1 we thus obtain the following property of pluripolar sets (compare with Theorem 5.8 in [4]):

COROLLARY. *If $E \subset \mathbb{C}^n$ is pluripolar then $E \subset \{u = -\infty\}$ for some $u \in \mathcal{D}(\mathbb{C}^n)$.*

The main tool in the proof will be the following characterization of the class $\mathcal{D}$ (see [3]): for a negative plurisubharmonic function u we have $u \in \mathcal{D}$ if and only if there exists a sequence (or equivalently: for every sequence) $u_j \in PSH \cap C^\infty$ decreasing to u the sequences

$$(2) \qquad (-u_j)^{n-2-k} du_j \wedge d^c u_j \wedge (dd^c u_j)^k \wedge \omega^{n-1-k}, \quad k = 0, 1, \dots, n-2,$$

are locally uniformly weakly bounded (here $\omega := dd^c|z|^2$).

It follows easily from (2) that (1) is an optimal condition: if $\chi(\log|z_1|) \in \mathcal{D}$ then by (2) for $k = 0$ we have

$$\int_{\{|\zeta|<\varepsilon\}} \frac{(-\chi(\log|\zeta|))^{n-2}(\chi'(\log|\zeta|))^2}{|\zeta|^2} d\lambda(\zeta) < \infty,$$

which is equivalent to

$$\int_{-\infty}^{\log\varepsilon} (-\chi(t))^{n-2}(\chi'(t))^2 dt < \infty.$$

A result related to Theorem 1 has been proved by Bedford and Taylor (see [1], p. 66-69). They namely showed the following:

THEOREM 2. *Assume that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing and such that*

$$\int_1^\infty \frac{\phi(x)}{x} dx < \infty.$$

Let v be a plurisubharmonic function such that for some negative plurisubharmonic u we have $-(-u\,\phi\circ u)^{1/n} \le v$. Then $v \in \mathcal{D}$.

We will now show how Theorem 1 implies Theorem 2. Set

$$\gamma(t) := -\frac{1}{2}\int_t^0 \sqrt{\frac{\phi(-s)}{-s}}\, ds, \quad t \le 0.$$

Then

$$\gamma'(t) = \frac{1}{2}\sqrt{\frac{\phi(-t)}{-t}}$$

and thus $\gamma : \mathbb{R}_- \longrightarrow \mathbb{R}_-$ is convex and increasing. Moreover,

$$\frac{d}{dt}\left(-(-t\phi(-t))^{1/2}\right) = \frac{1}{2}\left(-(-t\phi(-t))^{1/2}\right)^{-1/2}(\phi(-t) - t\phi'(-t)) \le \gamma'(t).$$

Therefore $\gamma(t) \le -(-t\phi(-t))^{1/2}$, $t \le 0$. Thus

$$\chi(t) := -(-\gamma(t))^{2/n} \le -(-t\phi(-t))^{1/n},$$

$\chi : \mathbb{R}_- \longrightarrow \mathbb{R}_-$ is convex and increasing, and

$$\int_{-\infty}^{-1}(-\chi(t))^{n-2}(\chi'(t))^2 dt = \frac{4}{n^2}\int_{-\infty}^{-1}(\gamma'(t))^2 dt < \infty.$$

By Theorem 1 we have $\chi \circ u \in \mathcal{D}$ and it is now enough to apply Theorem 1.2 in [3].

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Proof

Theorem 1 will be proved by successive application of the following estimates:

Lemma. *Let $\gamma : \mathbb{R} \longrightarrow \mathbb{R}_+$ be continuous and such that $\int_{-\infty}^{0} \gamma(t)\,dt < \infty$. Set*

$$f(t) := \int_{-\infty}^{t} \gamma(s)\,ds, \qquad g(t) := \int_{t}^{0} f(s)\,ds, \qquad t < 0,$$

so that $f, g \geq 0$, $f' = \gamma$, $g' = -f$. Assume that $K \Subset \Omega$, where Ω is a domain in $\mathbb{C}^n$. Let T, S be closed positive currents in Ω of bidegree, respectively, $(n-1, n-1)$ and $(n-2, n-2)$. Then for any negative $u \in PSH \cap C^{\infty}(\Omega)$ we have

$$\int_K \gamma \circ u \, du \wedge d^c u \wedge T \leq C_1 \int_\Omega g \circ u \, \omega \wedge T, \tag{3}$$

$$\int_K \gamma \circ u \, du \wedge d^c u \wedge dd^c u \wedge S \leq C_2 \int_\Omega f \circ u \, du \wedge d^c u \wedge \omega \wedge S, \tag{4}$$

where C_1, C_2 are positive constants depending only on K and Ω.

Proof. Let φ be a nonnegative test function in Ω with $\varphi = 1$ on K. Then

$$\begin{aligned}
\int_K \gamma \circ u \, du \wedge d^c u \wedge T &\leq \int_\Omega \varphi \gamma \circ u \, du \wedge d^c u \wedge T \\
&= \int_\Omega \varphi \, d(f \circ u) \wedge d^c u \wedge T \\
&= -\int_\Omega \varphi \, f \circ u \, dd^c u \wedge T - \int_\Omega f \circ u \, d\varphi \wedge d^c u \wedge T \\
&\leq -\int_\Omega f \circ u \, d\varphi \wedge d^c u \wedge T \\
&= \int_\Omega d\varphi \wedge d^c (g \circ u) \wedge T \\
&= -\int_\Omega g \circ u \, dd^c \varphi \wedge T \\
&\leq C_1 \int_\Omega g \circ u \, \omega \wedge T.
\end{aligned}$$

To show (4) we start the same way:

$$\begin{aligned}
\int_K \gamma \circ u \, du \wedge d^c u \wedge dd^c u \wedge S &\leq -\int_\Omega g \circ u \, dd^c \varphi \wedge dd^c u \wedge S \\
&= -\int_\Omega f \circ u \, du \wedge d^c u \wedge dd^c \varphi \wedge S \\
&\leq C_2 \int_\Omega f \circ u \, du \wedge d^c u \wedge \omega \wedge S. \ \square
\end{aligned}$$

Proof of Theorem 1. Without loss of generality we may assume that $u \leq -1$ and $\chi(0) = 0$ (because subtracting a constant from χ does not change (1)). For $k = 0, 1, \dots, n-2$ we set $\gamma_k := (-\chi)^{n-2-k}(\chi')^{k+2}$. Our goal is to show that for $K \Subset \Omega \subset \mathbb{C}^n$ and $u \in PSH \cap C^\infty(\Omega)$, $u \leq -1$, the following estimate holds

$$\int_K \gamma_k \circ u \, du \wedge d^c u \wedge (dd^c u)^k \wedge \omega^{n-k-1} \leq C \int_{-\infty}^{-1} \gamma_0(t) dt \, ||u||_{L^1(\Omega)}, \tag{5}$$

where C is a positive constant depending only on K an Ω. In view of (2) this will finish the proof.

By $\mathcal{F}$ denote the class of those γ that satisfy the assumptions of Lemma, that is $\gamma : \mathbb{R} \longrightarrow \mathbb{R}_+$ is continuous and $\int_{-\infty}^{-1} \gamma(t) dt < \infty$. For $\gamma \in \mathcal{F}$ we also define

$$(F\gamma)(t) := \int_{-\infty}^{t} \gamma(s) ds, \quad t \in \mathbb{R},$$

and $F^l \gamma := F \dots F\gamma$. Note that since $\chi'(s) \leq \chi(s)/s$ for $s < 0$, we have $\gamma_k \in \mathcal{F}$ by (1). We claim that $F\gamma_k \in \mathcal{F}$ for $k \geq 1$. For $a < 0$ by the Fubini theorem we have

$$(F^2\gamma_k)(a) = \int_{-\infty}^{a} \int_{-\infty}^{t} \gamma_k(s) \, ds \, dt = \int_{-\infty}^{a} \int_{s}^{a} \gamma_k(s) \, dt \, ds \leq -\int_{-\infty}^{a} s\gamma_k(s) \, ds.$$

Hence it follows that for $k = 1, \dots, n-2$

$$F^2 \gamma_k \leq F\gamma_{k-1} \quad \text{on } \mathbb{R}_-.$$

This implies that $F^l \gamma_k \in \mathcal{F}$, $l = 1, \dots, k+1$, and

$$F^{k+1}\gamma_k \leq (F\gamma_0)(-1) = \int_{-\infty}^{-1} \gamma_0(t) \, dt \quad \text{on } (-\infty, -1]. \tag{6}$$

Using (4) k times we will get

$$\int_K \gamma_k \circ u \, du \wedge d^c u \wedge (dd^c u)^k \wedge \omega^{n-k-1} \leq C(K, \Omega') \int_{\Omega'} (F^k \gamma_k) \circ u \, du \wedge d^c u \wedge \omega^{n-1},$$

where $K \Subset \Omega' \Subset \Omega$. Now set

$$g(t) := \int_t^0 (F^{k+1}\gamma_k)(s) \, ds, \quad t < 0.$$

Then by (3)

$$\int_K \gamma_k \circ u \, du \wedge d^c u \wedge (dd^c u)^k \wedge \omega^{n-k-1} \leq C(K, \Omega) \int_\Omega g \circ u \, \omega^n,$$

and by (6)

$$g(t) \leq |t| \int_{-\infty}^{-1} \gamma_0(s) \, ds, \quad t < 0.$$

We thus obtain (5). $\square$

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