Math. Z. 244, 153–161 (2003)
Mathematische Zeitschrift
DOI: 10.1007/s00209-002-0483-x

Regularity of the degenerate Monge-Ampère equation on compact Kähler manifolds

Zbigniew Błocki

Jagiellonian University, Institute of Mathematics, Reymonta 4, 30-059 Kraków, Poland
(e-mail blocki@im.uj.edu.pl)

Received: 18 April 2002 / Published online: 24 February 2003 –

Abstract. We study the $C^{1,1}$ and Lipschitz regularity of the solutions of the degenerate complex Monge-Ampère equation on compact Kähler manifolds. In particular, in view of the local regularity for the complex Monge-Ampère equation, the obtained $C^{1,1}$ regularity is a generalization of the Yau theorem which deals with the nondegenerate case.

1. Introduction

Let M be a compact Kähler manifold of the complex dimension n, $n \geq 2$, with the Kähler form ω. We will say that a function φ on M is admissible if it is upper semi-continuous, locally integrable and such that $dd^c\varphi + \omega \geq 0$, where $d = \partial + \bar\partial$ and $d^c = \sqrt{-1}(\bar\partial - \partial)$. The complex Monge-Ampère equation on M takes the form

$$(dd^c\varphi + \omega)^n = f\omega^n, \tag{1.1}$$

where $\omega^n = \omega \wedge \ldots \wedge \omega$. We shall normalize φ by

$$\int_M \varphi\omega^n = 0. \tag{1.2}$$

A necessary condition for f is

$$\int_M f\omega^n = \int_M \omega^n. \tag{1.3}$$

Partially supported by KBN Grant #2 P03A 028 19

In his famous paper [Y] Yau proved that for every positive C^∞ function f on M satisfying (1.3) there exists the unique C^∞ admissible φ satisfying (1.1) and (1.2) (the uniqueness had been earlier shown by Calabi). This result implies the Calabi conjecture.

The degenerate equation was recently studied by Kołodziej in [K1] and [K2]. By [BT] it is known that the left hand-side of (1.1) makes sense as a nonnegative Borel measure for bounded admissible φ. Throughout the rest of the paper we will assume that f is a nonnegative function on M satisfying (1.3) which belongs to $L^q(M)$ for some $q > 1$ and that φ is a continuous admissible solution of (1.1)-(1.2). The reason is that in this case, as shown in [K1], such a solution indeed exists and by [K2] (see also [B2]) it is in fact unique among all locally bounded admissible functions on M.

In this paper we study the regularity of (1.1) in the degenerate case. We will say that φ is almost $C^{1,1}$ if $\Delta\varphi$ is bounded. This is equivalent to the fact that the mixed complex derivatives $\varphi_{i\bar{j}} = \partial^2\varphi/\partial z_i \partial \bar{z}_j$ are bounded for $i, j = 1, \ldots, n$. Every almost $C^{1,1}$ function belongs to $W^{2,p}$ for all $p < \infty$, and thus to $C^{1,\alpha}$ for all $\alpha < 1$. But it does not necessarily belong to $W^{2,\infty} = C^{1,1}$.

Our main result is the following:

Theorem 1.1. *If $f^{1/(n-1)}$ is $C^{1,1}$ then φ is almost $C^{1,1}$. Moreover, we have*

$$\sup_M |\Delta\varphi| \le C,$$

where C depends only on M and on an upper bound for $||f^{1/(n-1)}||_{C^{1,1}}$.

The exponent $1/(n-1)$ appears naturally in the study of the real degenerate Monge-Ampère equation, see [G] and [GTW]. Theorem 1.1 generalizes the Yau theorem in view of the local regularity of the complex Monge-Ampère equation (see [B1, Theorem 2.6]).

We also get a result on Lipschitz regularity of the solutions of (1.1). However, we have been able to prove it only if either M has nonnegative bisectional curvature or φ is closed to a constant in the L^∞ norm:

Theorem 1.2. *Assume that M has nonnegative bisectional curvature. If $f^{1/n}$ is Lipschitz continuous then so is φ. Moreover,*

$$\sup_M |D\varphi| \le C,$$

where C depends only on M and on upper bound for $||f^{1/n}||_{C^{0,1}}$.

Theorem 1.3. *There exists a positive constant δ depending only on M such that if $||\varphi||_{L^\infty} \le \delta$ and $f^{1/n}$ is Lipschitz continuous then φ is Lipschitz continuous. Moreover,*

$$\sup_M |D\varphi| \le C,$$

where C depends only on M and on an upper bound for $||f^{1/n}||_{C^{0,1}}$.

By [K2, Corollary 4.4] it follows that there exists another positive constant δ' depending only on M such that if $||f-1||_{L^1} \le \delta'$ then $||\varphi||_{L^\infty} \le \delta$.

The proofs of Theorems 1.1-1.3 will proceed as follows. First, smoothing f in a right way and using the stability result from [K2] (or another one from [B2]) we reduce the problem to proving the estimate in the case when $f > 0$ and f is C^∞. In such a case the Yau theorem implies that φ must be C^∞. When showing these estimates we will use the following L^∞-estimate for the solutions of (1.1)-(1.2) (see [K1] or [T, p. 49-51]):

$$||\varphi||_{L^\infty} \le C, \tag{1.4}$$

where C depends only on M and on $||f||_{L^\infty}$.

2. Preliminaries

By $c_1, c_2, \ldots$ we will denote positive constants depending only on M. Since $d\omega = 0$, it follows that locally there exists a smooth plurisubharmonic function g with $\omega = dd^c g = 2\sqrt{-1}\partial\bar\partial g$. Then

$$\omega = 2\sum_{i,j} g_{i\bar j}\sqrt{-1}dz_i \wedge d\bar z_j.$$

We can find a finite number of coordinate systems covering M where such a g exists,

$$(g_{i\bar j}) \ge \frac{1}{c_1}(\delta_{ij}) \tag{2.1}$$

and

$$||g||_{C^{3,1}} = \sum_{0\le k\le 4} ||D^k g||_{L^\infty} \le c_2. \tag{2.2}$$

In what follows we will only use this finite number of charts and there we will always choose orthonormal coordinates, so that in particular the inequalities (2.1) and (2.2) will not be affected.

The condition that φ is admissible reads that the function

$$u := \varphi + g$$

is plurisubharmonic. We will say that φ is strongly admissible if there exists $\varepsilon > 0$ such that $dd^c\varphi + \omega \ge \varepsilon\omega$. This is of course equivalent to the fact that u is strongly plurisubharmonic. If φ is in addition smooth then the equation (1.1) takes the form

$$\det(u_{i\bar j}) = f\ \det(g_{i\bar j}).$$

Differentiating it with respect to z_p and $\bar{z}_q$ gives

$$u^{i\bar{j}}u_{i\bar{j}p} = (\log f)_p + (\log\det(g_{i\bar{j}}))_p, \tag{2.3}$$

$$u^{i\bar{j}}u_{i\bar{j}p\bar{q}} = (\log f)_{p\bar{q}} + (\log\det(g_{i\bar{j}}))_{p\bar{q}} + u^{i\bar{l}}u^{k\bar{j}}u_{i\bar{j}p}u_{k\bar{l}\bar{q}}, \tag{2.4}$$

where $(u^{i\bar{j}})$ denotes the inverse transposed matrix of $(u_{i\bar{j}})$.

When proving an a priori estimate by $C_1, C_2, \ldots$ we will denote constants that depend only on the desired quantities and will say that they are under control.

3. The $C^{1,1}$ regularity

Proof of Theorem 1.1. The partition of unity gives a finite number of smooth functions $\{\gamma^k\}$ on M such that $\sum_k \gamma_k = 1$, $0 \le \gamma_k \le 1$, and the support of every γ_k is contained in a chart. For $\varepsilon > 0$ set

$$g_\varepsilon := \sum_k \left(\gamma_k f^{1/(n-1)}\right) * \rho_\varepsilon^k + \varepsilon,$$

where ρ_ε^k is a standard regularizing kernel in a chart. We can find suitable constants μ_ε such that the functions

$$f_\varepsilon := \mu_\varepsilon g_\varepsilon^{n-1}$$

are positive, C^∞, satisfy (1.2), tend uniformly to f and

$$||f_\varepsilon^{1/(n-1)}||_{C^{1,1}} \le C_1.$$

Now, if φ_ε are the corresponding solutions of (1.1) given by the Yau theorem, then [K2, Corollary 4.4] implies that $\varphi_\varepsilon \to \varphi$ uniformly as $\varepsilon \to 0$ (by [B2, Theorem 3] we have the convergence in $L^{2n/(n-1)}$ which is also sufficient). We may therefore assume that φ is C^∞ and strongly admissible.

Note that

$$\Delta\varphi = \frac{1}{2}g^{i\bar{j}}\varphi_{i\bar{j}} > -\frac{n}{2}$$

and it is therefore enough to estimate $\Delta\varphi$ from above. Denote $G := (g_{i\bar{j}})$, $U := (u_{i\bar{j}})$ and set

$$V := G^{-1/2}UG^{-1/2}.$$

Then V is a positive hermitian matrix and one can easily show that the eigenvalues of V do not depend on the choice of holomorphic coordinates and thus they are the same in every chart. By $\lambda_{max}(V)$ denote the maximal eigenvalue of V. Set

$$\alpha := \log\lambda_{max}(V) - \varphi.$$

The function α is continuous on M and thus attains maximum at some $O \in M$. We have

$$\begin{aligned}\lambda_{max}(V) &= \max\left\{\overline{\zeta}^T V\zeta : \zeta \in \mathbb{C}^n, |\zeta| = 1\right\}\\ &= \max\left\{\overline{\zeta}^T U\zeta : \zeta \in \mathbb{C}^n, \overline{\zeta}^T G\zeta = |G^{1/2}\zeta|^2 = 1\right\}\\ &= \max\left\{\frac{u_{\zeta\bar{\zeta}}}{g_{\zeta\bar{\zeta}}} : \zeta \in \mathbb{C}^n \setminus \{0\}\right\},\end{aligned}$$

where $u_{\zeta\bar{\zeta}} = \overline{\zeta}^T U\zeta = \sum_{i,j} u_{i\bar{j}}\zeta_i\overline{\zeta}_j$. There exists $w \in \mathbb{C}^n$ with $|w| = 1$ such that

$$\lambda_{max}(V(O)) = \frac{u_{w\bar{w}}(O)}{g_{w\bar{w}}(O)}.$$

We may assume that at O U is diagonal and $u_{1\bar{1}} \geq u_{2\bar{2}} \geq \ldots \geq u_{n\bar{n}}$.

In a neighborhood of O define

$$\widetilde{\alpha} := \log\frac{u_{w\bar{w}}}{g_{w\bar{w}}} - A\varphi,$$

where A will be specified later. We have $\widetilde{\alpha} \leq \alpha \leq \alpha(O) = \widetilde{\alpha}(O)$, so that $\widetilde{\alpha}$ also has a maximum at O and thus for $p = 1, \ldots, n$ we have there

$$0 \geq \widetilde{\alpha}_{p\bar{p}} = \frac{u_{w\bar{w}p\bar{p}}}{u_{w\bar{w}}} - \frac{|u_{w\bar{w}p}|^2}{u_{w\bar{w}}^2} - \frac{g_{w\bar{w}p\bar{p}}}{g_{w\bar{w}}} + \frac{|g_{w\bar{w}p}|^2}{g_{w\bar{w}}^2} - A\varphi_{p\bar{p}}.$$

Hence by (2.4)

$$\begin{aligned}0 \geq \sum_p \frac{\widetilde{\alpha}_{p\bar{p}}}{u_{p\bar{p}}} &= \frac{(\log f)_{w\bar{w}}}{u_{w\bar{w}}} + \frac{(\log\det(g_{p\bar{q}}))_{w\bar{w}}}{u_{w\bar{w}}}\\ &+\frac{1}{u_{w\bar{w}}}\sum_{p,q}\frac{|u_{wp\bar{q}}|^2}{u_{p\bar{p}}u_{q\bar{q}}} - \frac{1}{u_{w\bar{w}}^2}\sum_p\frac{|u_{w\bar{w}p}|^2}{u_{p\bar{p}}}\\ &-\frac{1}{g_{w\bar{w}}}\sum_p\frac{g_{w\bar{w}p\bar{p}}}{u_{p\bar{p}}} + \frac{1}{g_{w\bar{w}}^2}\sum_p\frac{|g_{w\bar{w}p}|^2}{u_{p\bar{p}}} + A\sum_p\frac{g_{p\bar{p}}}{u_{p\bar{p}}} - nA.\end{aligned}$$

We shall now use an idea from the proof of [GTW, Lemma 2.1]. We will need an elementary lemma:

Lemma 3.1. *Let Ω be a domain in $\mathbb{R}^m$ and let $\psi \in C^{1,1}(\overline{\Omega})$ be nonnegative. Then $\sqrt{\psi} \in C^{0,1}(\Omega)$ and*

$$|(D\sqrt{\psi})(x)| \leq \max\left\{\frac{|D\psi(x)|}{2\mathrm{dist}(x,\partial\Omega)}, \frac{1+\sup_\Omega \lambda_{max}(D^2\psi)}{2}\right\}$$

for almost all $x \in \Omega$.

Proof. We may assume that $\psi > 0$ and that g is smooth. Set $r := \mathrm{dist}(x, \partial\Omega)$. If $\psi(x) \geq r^2$ then

$$|(D\sqrt{\psi})(x)| = \frac{|D\psi(x)|}{2\sqrt{\psi(x)}} \leq \frac{|D\psi(x)|}{2r}.$$

We may thus assume that $\psi(x) \leq r^2$. For fixed $\zeta \in \mathbb{R}^m$ with $|\zeta| = 1$ and $t \in \mathbb{R}$ with $|t| \leq \sqrt{r}$ set $h(t) := \psi(x + t\zeta)$. We may assume that $h'(0) \leq 0$ (otherwise consider $-\zeta$ instead of ζ). Set $y := \sqrt{\psi(x)} = \sqrt{h(0)}$. Then

$$0 < h(y) = y^2 + \int_0^y h'(t)dt.$$

We can thus find $t \in [0, y]$ such that $h'(t) \geq -h(0)/y = -y$. There exists $s \in [0, t]$ with

$$h''(s) = \frac{h'(t) - h'(0)}{t} \geq -1 - \frac{h'(0)}{y}.$$

Therefore

$$|(D_\zeta\sqrt{\psi})(x)| = \frac{|h'(0)|}{2\sqrt{h(0)}} \leq \frac{1 + h''(s)}{2}$$

and the lemma follows. □

Remark. One cannot expect that $\sqrt{\psi} \in C^{0,1}(\overline{\Omega})$ in the assertion of Lemma 3.1. For let for example Ω be the interval $(0, 1)$ in $\mathbb{R}$ and $\psi(x) = x$.

Proof of Theorem 1.1 continued. Denoting $\widetilde{f} := f^{1/(n-1)}$, by Lemma 3.1 we get

$$(\log f)_{w\bar{w}} = (n-1)\left(\frac{\widetilde{f}_{w\bar{w}}}{\widetilde{f}} - \frac{|\widetilde{f}_w|^2}{\widetilde{f}^2}\right) \geq -\frac{C_2}{\widetilde{f}}. \tag{3.1}$$

By (2.1) and (2.2)

$$\begin{aligned} &\frac{(\log\det(g_{p\bar{q}}))_{w\bar{w}}}{u_{w\bar{w}}} - \frac{1}{g_{w\bar{w}}}\sum_p \frac{g_{w\bar{w}p\bar{p}}}{u_{p\bar{p}}} + A\sum_p \frac{g_{p\bar{p}}}{u_{p\bar{p}}} \\ &\geq -\frac{c_3}{u_{w\bar{w}}} + (-c_4 + \frac{A}{c_1})\sum_p \frac{1}{u_{p\bar{p}}} = -\frac{c_3}{u_{w\bar{w}}} + c_4\sum_p \frac{1}{u_{p\bar{p}}} \end{aligned} \tag{3.2}$$

if we choose $A := 2c_1c_4$. From (2.1), the inequality between geometric and arithmetic means and since $u_{w\bar{w}} \leq u_{1\bar{1}}$ at O, we also obtain

$$c_4\sum_p \frac{1}{u_{p\bar{p}}} \geq c_4\frac{n-1}{(u_{2\bar{2}}\dots u_{n\bar{n}})^{1/(n-1)}} \geq \frac{u_{w\bar{w}}^{1/(n-1)}}{c_5\widetilde{f}} \tag{3.3}$$

The Schwartz inequality for every p gives

$$|u_{w\bar{w}p}|^2 \leq u_{w\bar{w}} \sum_q \frac{|u_{wp\bar{q}}|^2}{u_{q\bar{q}}},$$

thus

$$\frac{1}{u_{w\bar{w}}} \sum_{p,q} \frac{|u_{wp\bar{q}}|^2}{u_{p\bar{p}} u_{q\bar{q}}} - \frac{1}{u_{w\bar{w}}^2} \sum_p \frac{|u_{w\bar{w}p}|^2}{u_{p\bar{p}}} \geq 0. \tag{3.4}$$

Combining (3.2)–(3.5) and multiplying (3.1) by $c_5 \widetilde{f} u_{w\bar{w}}$ we get

$$u_{w\bar{w}}^{n/(n-1)} - C_3 u_{w\bar{w}} - C_4 \leq 0$$

at O. Therefore $u_{w\bar{w}} \leq C_5$ at O and by (1.4)

$$\max_M \alpha \leq \widetilde{\alpha}(O) \leq C_6$$

from which the theorem easily follows. □

4. The Lipschitz regularity

Proof of Theorem 1.3. By a similar argument as at the beginning of the proof of Theorem 1.1 we may assume that φ is C^∞ and strongly admissible. Set $s := ||\varphi||_{L^\infty}$ and

$$\alpha := \frac{\beta^a}{\varphi + 2s},$$

where

$$\beta := |D\varphi|^2 = g^{i\bar{j}} \varphi_i \varphi_{\bar{j}}$$

and $a > 1$ will be specified later. The function α attains its maximum at some $O \in M$. We may assume that the matrix $(u_{i\bar{j}})$ is diagonal at O. At O we have for $p = 1, \dots, n$

$$0 = \alpha_p = \frac{a\, \beta^{a-1} \beta_p}{\varphi + 2s} - \frac{\beta^a \varphi_p}{(\varphi + 2s)^2}, \tag{4.1}$$

and, by (4.1),

$$0 \geq \alpha_{p\bar{p}} = \frac{a\, \beta^{a-1} \beta_{p\bar{p}}}{\varphi + 2s} - \frac{\beta^a \varphi_{p\bar{p}}}{(\varphi + 2s)^2} + \frac{(a-1)\beta^a |\varphi_p|^2}{a(\varphi + 2s)^3}. \tag{4.2}$$

We have

$$\begin{aligned} \beta_{p\bar{p}} &= (g^{i\bar{j}})_{p\bar{p}} \varphi_i \varphi_{\bar{j}} + 2\, Re((g^{i\bar{j}})_p \varphi_{i\bar{p}} \varphi_{\bar{j}}) + 2\, Re((g^{i\bar{j}})_p \varphi_i \varphi_{\bar{j}\bar{p}}) \\ &\qquad + 2\, Re(g^{i\bar{j}} \varphi_{ip\bar{p}} \varphi_{\bar{j}}) + g^{i\bar{j}} \varphi_{i\bar{p}} \varphi_{\bar{j}p} + g^{i\bar{j}} \varphi_{ip} \varphi_{\bar{j}\bar{p}} \\ &\geq -c_6 \beta + 2\, Re(g^{i\bar{j}} \varphi_{ip\bar{p}} \varphi_{\bar{j}}) + \frac{1}{c_7} g^{i\bar{j}} \varphi_{i\bar{p}} \varphi_{\bar{j}p} \end{aligned} \tag{4.3}$$

by (2.1) and (2.2). From (2.3) we get

$$\sum_p \frac{2\,Re(g^{i\bar{j}}\varphi_{ip\bar{p}}\varphi_{\bar{j}})}{u_{p\bar{p}}} \geq -c_8\sqrt{\beta}\left(1+|\nabla(\log f)|+\sum_p\frac{1}{u_{p\bar{p}}}\right). \tag{4.4}$$

Note that, by the inequality between arithmetic and geometric means,

$$|D(\log f)| \leq |D(f^{1/n})|\sum_p \frac{1}{u_{p\bar{p}}}. \tag{4.5}$$

Moreover,

$$\sum_p \frac{g^{i\bar{j}}\varphi_{i\bar{p}}\varphi_{\bar{j}p}}{u_{p\bar{p}}} = \sum_p\left(g^{p\bar{p}}u_{p\bar{p}}+\frac{g_{p\bar{p}}}{u_{p\bar{p}}}\right)-2n \geq \frac{1}{c_9}\Delta u - 2n. \tag{4.6}$$

We also have

$$-\sum_p \frac{\varphi_{p\bar{p}}}{u_{p\bar{p}}} \geq \frac{1}{c_1}\sum_p\frac{1}{u_{p\bar{p}}} - n \tag{4.7}$$

and

$$\sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} \geq \frac{\beta}{c_{10}\Delta u}. \tag{4.8}$$

Since

$$\frac{\Delta u}{c_7c_9}+\frac{(a-1)\beta^2}{c_{10}a^2(\varphi+2s)^2\Delta u} \geq \frac{\sqrt{a-1}\,\beta}{c_{11}a(\varphi+2s)}. \tag{4.9}$$

Combining (4.2)–(4.9) we get

$$\begin{aligned} 0 &\geq \frac{\varphi+2s}{a\,\beta^{a-1}}\sum_p\frac{\alpha_{p\bar{p}}}{u_{p\bar{p}}} \\ &\geq -c_6\beta\sum_p\frac{1}{u_{p\bar{p}}} - c_8\sqrt{\beta}\left(1+C_1\sum_p\frac{1}{u_{p\bar{p}}}\right)+\frac{\sqrt{a-1}\,\beta}{c_{11}a(\varphi+2s)} \\ &\quad -\frac{2n}{c_7}+\frac{\beta}{a(\varphi+2s)}\left(\frac{1}{c_1}\sum_p\frac{1}{u_{p\bar{p}}}-n\right). \end{aligned} \tag{4.10}$$

If we now choose a so that

$$\frac{\sqrt{a-1}}{c_{11}} = 2n$$

and δ so small that

$$\frac{1}{3\delta\,a\,c_1} - c_6 \geq 1$$

then

$$(\beta - c_8 C_1\sqrt{\beta})\sum_p \frac{1}{u_{p\bar{p}}} + \frac{\beta}{C_2} - c_8\sqrt{\beta} - \frac{2n}{c_7} \le 0.$$

Therefore $\beta \le C_3$ at O and the theorem follows. □

Proof of Theorem 1.2. The proof is almost the same to the proof of Theorem 1.3. To improve (4.3) we compute

$$\begin{aligned}(g^{i\bar{j}})_{p\bar{p}} &= -g^{i\bar{l}}g^{k\bar{j}}g_{k\bar{l}p\bar{p}} + g^{i\bar{t}}g^{s\bar{l}}g^{k\bar{j}}g_{k\bar{l}p}g_{s\bar{t}\bar{p}} + g^{i\bar{l}}g^{k\bar{t}}g^{s\bar{j}}g_{k\bar{l}p}g_{s\bar{t}\bar{p}} \\ &= g^{i\bar{l}}g^{k\bar{j}}R_{k\bar{l}p\bar{p}} + g^{i\bar{l}}g^{k\bar{t}}g^{s\bar{j}}g_{k\bar{l}p}g_{s\bar{t}\bar{p}}.\end{aligned}$$

Therefore the nonnegative bisectional curvature implies that

$$(g^{i\bar{j}})_{p\bar{p}}\varphi_i\varphi_{\bar{j}} \ge 0$$

and we may assume that $c_6 = 0$ in (4.3) and thus also in (4.10). By (1.4) we have $s \le C_4$. Hence

$$\left(\frac{\beta}{3c_1 C_4 a} - c_8 C_1\sqrt{\beta}\right)\sum_p \frac{1}{u_{p\bar{p}}} + \frac{\sqrt{a-1}\beta}{3c_{11}C_4 a} - c_8\sqrt{\beta} - \frac{2n}{c_7} \le 0.$$

It now suffices to choose an arbitrary $a > 1$ to get the required estimate. □

Acknowledgements. This paper was written during the author's stay at the Max-Planck-Institut für Mathematik in Bonn. He is grateful for the hospitality he received there.

References

[BT] Bedford, E., Taylor, B.A.: A new capacity for plurisubharmonic functions. Acta Math. **149**, 1–41 (1982)

[B1] Błocki, Z.: On the regularity of the complex Monge-Ampère operator. In: Complex geometric analysis in Pohang (1997). Contemp. Math. **222**, 181–189, Amer. Math. Soc. (1999)

[B2] Błocki, Z.: Uniqueness and stability for the complex Monge-Ampère equation on compact Kähler manifolds. Preprint 2002

[G] Guan, P.: C^2 a priori estimates for degenerate Monge-Ampère equations. Duke Math. J. **86**, 323–346 (1997)

[GTW] Guan, P., Trudinger, N.S., Wang, X.-J.: On the Dirichlet problem for degenerate Monge-Ampère equations. Acta Math. **182**, 87–104 (1999)

[K1] Kołodziej, S.: The complex Monge-Ampère equation. Acta Math. **180**, 69–117 (1998)

[K2] Kołodziej, S.: Stability of solutions to the complex Monge-Ampère equation on compact Kähler manifolds. IMUJ Preprint 2001/07, http://www.im.uj.edu.pl/preprint, to appear in Indiana Univ. Math. J

[T] Tian, G.: Canonical metrics in Kähler geometry. Birkhäuser 2000

[Y] Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I. Comm. Pure Appl. Math. **31**, 339–411 (1978)