A note on the Hörmander, Donnelly-Fefferman, and Berndtsson
$L^2$-estimates for the $\bar{\partial}$-operator

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Abstract. We give upper and lower bounds for constants appearing in the $L^2$-
estimates for the $\bar{\partial}$-operator due to Donnelly–Fefferman and Berndtsson.

1. Introduction. Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^n$ and suppose that a form
\[ \alpha = \sum_{j=1}^{n} \alpha_j dz_j \in L^2_{\text{loc},(0,1)}(\Omega) \]
is $\bar{\partial}$-closed (that is, $\bar{\partial} \alpha = 0$, which means that $\partial \alpha_j / \partial z_k = \partial \alpha_k / \partial z_j$, $j, k = 1, \ldots, n$). The equation
\[ \bar{\partial} u = \alpha \]
(which is equivalent to the system of equations $\partial u / \partial z_j = \alpha_j$, $j = 1, \ldots, n$) always has a solution $u \in L^2_{\text{loc},(0,1)}$ and the difference of any two solutions of (1) is a holomorphic function in $\Omega$ (see [6]). A slight modification of the proof of Hörmander’s estimate [6, Lemma 4.4.1] (see e.g. [4, Théorème 4.1]) shows that for every smooth, strongly plurisubharmonic function $\varphi$ in $\Omega$ we can find a solution to (1) satisfying
\[ \int_{\Omega} |u|^2 e^{-\varphi} \, d\lambda \leq \int_{\Omega} |\alpha|^2_{i \partial \bar{\partial} \varphi} e^{-\varphi} \, d\lambda. \]
By $|\alpha|_{i \partial \bar{\partial} \varphi}$ we understand the pointwise norm of $\alpha$ with respect to the Kähler metric $i \partial \bar{\partial} \varphi$, that is,
\[ |\alpha|^2_{i \partial \bar{\partial} \varphi} = \sum_{j,k=1}^{n} \varphi^{j} \bar{\alpha}_j \alpha_k, \]

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where \((\varphi^j^k)\) is the inverse transposed matrix of \(\left(\partial^2 \varphi / \partial z_j \partial z_k\right)\). The function \(|\alpha|^2_{i\partial \bar{\partial} \varphi}\) is the least function \(H\) satisfying

\[
\iota \alpha \wedge \alpha \leq H i \partial \bar{\partial} \varphi,
\]

and one can obtain the estimate \((H)\) for an arbitrary plurisubharmonic function \(\varphi\) in \(\Omega\), where instead of \(|\alpha|^2_{i\partial \bar{\partial} \varphi}\) we take a function \(H\) satisfying (2) (see [3] for the approximation argument based on the proof of [6, Theorem 4.4.2]).

A very useful variation of the Hörmander estimate \((H)\) was proved by Donnelly and Fefferman [5]. Let in addition \(\psi\) be a plurisubharmonic function in \(\Omega\) satisfying

\[
i \partial \psi \wedge \bar{\partial} \psi \leq i \partial \bar{\partial} \psi.
\]

This is equivalent to the fact that the function \(-e^{-\psi}\) is plurisubharmonic, that is,

\[\psi = -\log(-v)\]

for a certain negative plurisubharmonic function \(v\) in \(\Omega\). Then one can find a solution to (1) with

\[
(DF) \quad \int_{\Omega} |u|^2 e^{-\varphi} \, d\lambda \leq C \int_{\Omega} |\alpha|^2_{i\partial \bar{\partial} \psi} e^{-\varphi} \, d\lambda,
\]

where \(C\) is an absolute constant.

Berndtsson [1] showed that for any \(\delta\) with \(0 < \delta < 1\) one can find a solution to (1) with

\[
(B) \quad \int_{\Omega} |u|^2 e^{-\varphi + \delta \psi} \, d\lambda \leq \frac{4}{\delta(1-\delta)^2} \int_{\Omega} |\alpha|^2_{i\partial \bar{\partial} \psi} e^{-\varphi + \delta \psi} \, d\lambda,
\]

where \(\varphi\) and \(\psi\) are as above. The Berndtsson estimate easily implies the Donnelly–Fefferman estimate—it is enough to consider the function \(\varphi + \delta \psi\) instead of \(\varphi\). The best choice for \(\delta\) is then \(\delta = 1/3\), one then gets \(C = 27\) in the Donnelly–Fefferman estimate. In [2] Berndtsson showed that the estimate \((B)\) follows easily from the Hörmander estimate \((H)\). Using his arguments it was shown in [3] that the constant in the Berndtsson estimate can be improved to \(1/\delta(1-\sqrt{\delta})^2\). From this with \(\delta = 1/4\) one gets \(C = 16\) in \((DF)\).

By \(C_B(\delta)\) denote the best constant in the Berndtsson estimate. Then \(C_{DF} = C_B(0)\) is the best constant in the Donnelly–Fefferman estimate. The goal of this note is to show the following result.

**Proposition.** We have

\[
\frac{4}{(1-\delta)(2-\delta)} \leq C_B(\delta) \leq \frac{4}{(1-\delta)^2}, \quad 0 \leq \delta < 1.
\]

**Corollary.** \(2 \leq C_{DF} \leq 4\).
Note that
\[ \frac{4}{(1 - \delta)^2} < \frac{1}{\delta(1 - \sqrt{\delta})^2} < \frac{4}{\delta(1 - \delta)^2}, \quad 0 < \delta < 1, \]
so the upper bound is an improvement of the constants from [1] and [3]. Concerning the lower bound, it was noted already in [1] that the best constant cannot be better than \( C/(1 - \delta) \), so that in particular the Berndtsson estimate does not hold for \( \delta = 1 \).

**Proofs.** Using the Berndtsson argument (see the proof of [2, Lemma 2.2]) we first prove the estimate

\[ (3) \quad \int_{\Omega} |u|^2 e^{-\varphi + \delta \psi} d\lambda \leq \frac{4}{(1 - \delta)^2} \int_{\Omega} He^{-\varphi + \delta \psi} d\lambda, \]

where \( i\bar{\alpha} \wedge \alpha \leq H i\bar{\partial} \partial \psi \), that is, the upper bound in the proposition. We will just choose the constants more carefully than in [2]. Due to the approximation argument from [3] we may assume that \( \Omega \) is bounded and \( \varphi, \psi \) are smooth and continuous up to the boundary. Then for any real \( a \) we have the equality of sets

\[ L^2(\Omega, e^{-\varphi - a\psi}) = L^2(\Omega). \]

Let \( u \) be the minimal solution to (1) in the \( L^2(\Omega, e^{-\varphi - a\psi}) \)-norm (\( a \) will be specified later). This means that \( u \) is perpendicular to the subspace \( H^2(\Omega) \) of square integrable holomorphic functions in \( \Omega \) in the Hilbert space \( L^2(\Omega, e^{-\varphi - a\psi}) \), that is,

\[ \int_{\Omega} u \bar{f} e^{-\varphi - a\psi} d\lambda = 0, \quad f \in H^2(\Omega). \]

Let \( v := e^{b\psi} u \), where \( b \in \mathbb{R} \) will be specified later. Then

\[ \int_{\Omega} v \bar{f} e^{-\varphi - (a+b)\psi} d\lambda = 0, \quad f \in H^2(\Omega). \]

This means that \( v \) is a minimal solution to the equation

\[ \bar{\partial} v = \beta \]

in the \( L^2(\Omega, e^{-\varphi - (a+b)\psi}) \)-norm, where

\[ \beta = \bar{\partial}(e^{b\psi} u) = e^{b\psi}(\alpha + bu \bar{\partial} \psi). \]

If \( P, Q \) are any \( (1, 0) \)-forms then for any \( t > 0 \) we have

\[
i(P + Q) \wedge (\bar{P} + \bar{Q})
= (1 + t)iP \wedge \bar{P} + (1 + t^{-1})iQ \wedge \bar{Q} - ti(P - t^{-1}Q) \wedge (\bar{P} - t^{-1}\bar{Q})
\leq (1 + t)iP \wedge \bar{P} + (1 + t^{-1})iQ \wedge \bar{Q}.\]
Therefore
\[ i\beta \wedge \beta \leq e^{2b\psi}[(1+t)\iota\bar{\alpha} \wedge \alpha + (1+t^{-1})b^2|u|^2i\bar{\partial}\psi \wedge \bar{\partial}\psi] \]
\[ \leq e^{2b\psi}[(1+t)H + (1+t^{-1})b^2|u|^2]i\partial\bar{\partial}\psi \]
\[ \leq \frac{e^{2b\psi}}{a+b}[(1+t)H + (1+t^{-1})b^2|u|^2]i\partial\bar{\partial}(\varphi + (a+b)\psi) \]

provided that \( a+b > 0 \). From the Hörmander estimate (H) applied to the form \( \beta \) and the function \( \varphi + (a+b)\psi \) we obtain
\[ \int \Omega |v|^2 e^{-\varphi-(a+b)\psi} d\lambda \leq \frac{1}{a+b} \int \Omega [(1+t)H + (1+t^{-1})b^2|u|^2]e^{-\varphi+(b-a)\psi} d\lambda. \]

Thus, taking \( b = a + \delta \), we get
\[ \int \Omega |u|^2 e^{-\varphi+\delta\psi} d\lambda \leq \frac{1+t}{2a+\delta} \int \Omega He^{-\varphi+\delta\psi} d\lambda \]
\[ + \frac{(1+t^{-1})(a+\delta)^2}{2a+\delta} \int \Omega |u|^2 e^{-\varphi+\delta\psi} d\lambda. \]

We now only have to minimize the positive values of the function
\[ \frac{1+t}{2a+\delta} = \frac{t(1+t)}{t(2a+\delta) - (1+t)(a+\delta)^2} \]
for \( t > 0 \) and \( a > -\delta/2 \). The minimum is easily shown to be attained for \( a = -\delta + t/(1+t) \) and \( t = (1+\delta)/(1-\delta) \) (then \( a = (1-\delta)/2 \)). For these values of \( a \) and \( t \) we obtain (3).

To get the lower bound in the proposition we will use the following lemma.

**Lemma.** Let \( \Omega = \Delta \) be the unit disc in \( \mathbb{C} \). Set \( \alpha = d\bar{z} \) and assume that \( F \) is a nonnegative, continuous, radially symmetric (that is, \( F(z) = \gamma(|z|) \)) function in \( \Delta \). Then the function \( u(z) = \bar{z} \) is the minimal solution to (1) in the \( L^2(\Delta,F) \)-norm (provided that \( u \) belongs to \( L^2(\Delta,F) \), that is, \( \int_0^1 r^3\gamma(r) dr < \infty \)).

**Proof.** We have to show that
\[ \int f\bar{u}F d\lambda = 0, \quad f \in \mathcal{O}(\Delta) \cap L^2(\Delta,F). \]

Write
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \Delta, \]
where the convergence is uniform on every circle in $\Delta$. Therefore
\[
\int f \overline{u} F \, d\lambda = 2\pi \sum_{n=0}^{\infty} a_n r^{n+2} \gamma(r) \int_0^{2\pi} e^{i(n+1)t} \, dt \, dr = 0. \]

We now consider the estimate (B) with $n = 1$, $\Omega = \Delta$, $\varphi = 0$ and $\psi(z) = -\log(-\log |z|)$. In this case the least value of the left-hand side of (B) is attained for $u(z) = \overline{z}$. Then
\[
\int_{\Delta} |u|^2 e^{-\varphi + \delta \psi} \, d\lambda = 2\pi \int_0^1 r^3 (-\log r)^{-\delta} \, dr
\]
and
\[
\int_{\Delta} |\alpha|^2 e^{-\varphi + \delta \psi} \, d\lambda = 8\pi \int_0^1 r^3 (-\log r)^{2-\delta} \, dr = \pi \frac{(2-\delta)(1-\delta)}{2} \int_0^1 r^3 (-\log r)^{-\delta} \, dr
\]
after double integration by parts.

References


