A Lower Bound for the Bergman Kernel and the Bourgain-Milman Inequality

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Abstract For pseudoconvex domains in $\mathbb{C}^n$ we prove a sharp lower bound for the Bergman kernel in terms of volume of sublevel sets of the pluricomplex Green function. For $n = 1$ it gives in particular another proof of the Suita conjecture. If Ω is convex then by Lempert's theory the estimate takes the form $K_\Omega(z) \geq 1/\lambda_{2n}(I_\Omega(z))$, where $I_\Omega(z)$ is the Kobayashi indicatrix at z. One can use this to simplify Nazarov's proof of the Bourgain-Milman inequality from convex analysis. Possible further applications of Lempert's theory in this area are also discussed.

1 Introduction

For a domain Ω in $\mathbb{C}^n$ and $w \in \Omega$ we are interested in the Bergman kernel

$$K_\Omega(w) = \sup\{|f(w)|^2 : f \in \mathcal{O}(\Omega), \int_\Omega |f|^2 d\lambda_{2n} \leq 1\}$$

and in the pluricomplex Green function with pole at w

$$G_{\Omega,w} = \sup\{u \in PSH^-(\Omega) : \limsup_{z\to w}(u(z) - \log|z-w|) < \infty\}$$

(Here PSH^- denotes the class of negative plurisubharmonic functions.)

Our main result is the following bound:

Theorem 1. *Assume that Ω is pseudoconvex. Then for $w \in \Omega$ and $a \geq 0$ we have*

$$K_\Omega(w) \geq \frac{1}{e^{2na}\lambda_{2n}(\{G_{\Omega,w} < -a\})}. \tag{1}$$

This estimate seems to be very accurate. It is certainly optimal in the sense that if Ω is a ball centered at w then we get equality in (1) for all a. It is useful and

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B. Klartag, E. Milman (eds.), *Geometric Aspects of Functional Analysis*,
Lecture Notes in Mathematics 2116, DOI 10.1007/978-3-319-09477-9_4

not trivial already for $n = 1$. Note that in this case if we let a tend to ∞ then we immediately obtain

$$K_\Omega \geq \frac{1}{\pi} c_\Omega^2, \tag{2}$$

where

$$c_\Omega(z) = \exp(\lim_{\zeta \to z}(G_{\Omega,z}(\zeta) - \log|\zeta - z|))$$

is the logarithmic capacity of $\mathbb{C} \setminus \Omega$ with respect to z. This is precisely the inequality conjectured by Suita [19] and recently proved in [5].

A lower bound of the Bergman kernel in terms of the volume of the sublevel sets of the Green function follows from an estimate of Herbort (Proposition 3.6 in [10] with $f \equiv 1$). The main point in (1) is that the constant is optimal. Our proof of (1) uses the L^2-estimate for the $\bar\partial$-equation of Donnelly-Fefferman [8] from which we can first get a weaker version:

$$K_\Omega(w) \geq \frac{c(n,a)}{\lambda_{2n}(\{G_{\Omega,w} < -a\})},$$

where

$$c(n,a) = \left(\frac{\mathrm{Ei}\,(na)}{\mathrm{Ei}\,(na) + 2}\right)^2$$

and

$$\mathrm{Ei}\,(b) = \int_b^\infty \frac{ds}{se^s} \tag{3}$$

(for $b > 0$). Then we employ the tensor power trick and use the fact that

$$\lim_{m\to\infty} c(nm,a)^{1/m} = e^{-2na}.$$

This way we get an optimal constant in (1).

Our new proof of the one-dimensional estimate (2) makes crucial use of many complex variables. The use of the tensor power trick here replaces a special ODE in [5]. It should be noted though that this works only for the Suita conjecture, we do not get the Ohsawa-Takegoshi extension theorem from Theorem 1.

It is probably interesting to investigate the limit of the right-hand side of (1) as a tends to ∞ also in higher dimensions. We suspect that it always exists, at least for sufficiently regular domains. This way we would get a certain counterpart of logarithmic capacity in higher dimensions. Using Lempert's theory [15, 16] one can check what happens with this limit for smooth and strongly convex domains, see Proposition 3 below. This way we get the following bound:

Theorem 2. *Let Ω be a convex domain in $\mathbb{C}^n$. Then for $w \in \Omega$*

$$K_\Omega(w) \geq \frac{1}{\lambda_{2n}(I_\Omega(w))},$$

where

$$I_\Omega(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}$$

is the Kobayashi indicatrix (here Δ denotes the unit disc).

One can use Theorem 2 to simplify Nazarov's approach [17] to the Bourgain-Milman inequality [6]. For a convex symmetric body (i.e. open, bounded) L in $\mathbb{R}^n$ its dual is given by

$$L' := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } x \in L\}.$$

The product $\lambda_n(L)\lambda_n(L')$ is called a Mahler volume of L. It is independent of linear transformations and on an inner product in $\mathbb{R}^n$, and thus depends only on the finite dimensional Banach space structure whose unit ball is L. The Blaschke-Santaló inequality says that the Mahler volume is maximized by balls.

On the other hand, the still open Mahler conjecture states that it is minimized by cubes. A partial result in this direction is the Bourgain-Milman inequality [6] which says that there exists $c > 0$ such that

$$\lambda_n(L)\lambda_n(L') \geq c^n \frac{4^n}{n!}. \tag{4}$$

The Mahler conjecture is equivalent to saying that we can take $c = 1$ in (4). Currently, the best known constant in (4) is $\pi/4$ and is due to Kuperberg [14].

Nazarov [17] recently proposed a complex-analytic approach to (4). He considered tube domain $T_L := L + i\mathbb{R}^n$ and proved the following bounds for the Bergman kernel at the origin:

$$K_{T_L}(0) \leq \frac{n!}{\pi^n} \frac{\lambda_n(L')}{\lambda_n(L)} \tag{5}$$

$$K_{T_L}(0) \geq \left(\frac{\pi}{4}\right)^{2n} \frac{1}{(\lambda_n(L))^2}. \tag{6}$$

This gave (4) with $c = (\pi/4)^3$. The upper bound (5) was obtained relatively easily from Rothaus' formula for the Bergman kernel in tube domains (see [18] and [12]):

$$K_{T_L}(0) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{d\lambda_n}{J_L},$$

where

$$J_L(y) = \int_L e^{-2x\cdot y} d\lambda_n(x).$$

For the lower bound (6) Nazarov used the Hörmander estimate [11] for $\bar\partial$.

We will show that (6) follows easily from Theorem 2. It should be noted however that although we are using the Donnelly-Fefferman estimate here, it can be deduced quite easily from the Hörmander estimate (see [1]), so the latter still plays a crucial role.

We conjecture that in fact the following lower bound holds:

$$K_{T_L}(0) \geq \left(\frac{\pi}{4}\right)^n \frac{1}{(\lambda_n(L))^2}. \tag{7}$$

Since we have equality for cubes, this would be optimal. In Sect. 4 we discuss possible applications of Lempert's theory to this problem.

The author learned about the Nazarov paper [17] from professor Vitali Milman during his visit to Tel Aviv in December 2011. He is also grateful to Semyon Alesker for his invitation and hospitality.

2 Proofs of Theorems 1 and 2

Proof of Theorem 1. By approximation we may assume that Ω is bounded and hyperconvex, so that by [7] the Green function $G := G_{\Omega,w}$ is continuous on $\bar\Omega \setminus \{w\}$. We may also assume that $a > 0$, as for $a = 0$ it is enough to take $f \equiv 1$ in the definition of the Bergman kernel. Set

$$\varphi := 2nG, \quad \psi := -\log(-G)$$

and

$$\alpha := \bar\partial(\chi \circ G) = (\chi' \circ G)\,\bar\partial G,$$

where χ will be determined later. We have

$$i\bar\alpha \wedge \alpha \leq (\chi' \circ G)^2 i\partial G \wedge \bar\partial G \leq G^2(\chi' \circ G)^2 i\partial\bar\partial\psi.$$

By the Donnelly-Fefferman estimate [8] (see also [1, 2], and [3] for a formulation with non-smooth weights which is needed here) we can find $u \in L^2_{loc}(\Omega)$ solving $\bar\partial u = \alpha$ and such that

$$\int_\Omega |u|^2 d\lambda_{2n} \le \int_\Omega |u|^2 e^{-\varphi} d\lambda_{2n} \le C \int_\Omega G^2(\chi' \circ G)^2 e^{-2nG} d\lambda_{2n},$$

where C is an absolute constant (in fact, the optimal one is $C = 4$, see [2, 4]). We now set

$$\chi(t) := \begin{cases} 0 & t \ge -a, \\ \int_a^{-t} \frac{e^{-ns}}{s}\, ds, & t < -a, \end{cases}$$

so that

$$\int_\Omega |u|^2 d\lambda_{2n} \le C\, \lambda_{2n}(\{G < -a\}).$$

The function $f := \chi \circ G - u$ is holomorphic and since $\chi \circ G$ is continuous, we see that u must be continuous. We also have $u(w) = 0$ because $e^{-\varphi}$ is not integrable near w (by monotonicity of the Green function we have $G_\Omega(z,w) \le \log|z-w| - \log r$ if $B(z,r) \subset \Omega$). Therefore

$$f(w) = \chi(-\infty) = \mathrm{Ei}\,(na)$$

with Ei given by (3). We also have (with $||\cdot||$ denoting the L^2-norm in Ω)

$$||f|| \le ||\chi \circ G|| + ||u|| \le (\chi(-\infty) + \sqrt{C})\sqrt{\lambda_{2n}(\{G < -a\})}.$$

Therefore

$$K_\Omega(w) \ge \frac{|f(w)|^2}{||f||^2} \ge \frac{c(n,a)}{\lambda_{2n}(\{G < -a\})},$$

where

$$c(n,a) = \frac{\mathrm{Ei}\,(na)^2}{(\mathrm{Ei}\,(na) + \sqrt{C})^2}.$$

We are now going to use the tensor power trick. For a big natural number m consider the domain $\widetilde{\Omega} = \Omega^m$ in $\mathbb{C}^{nm}$ and $\widetilde{w} = (w, \dots, w) \in \widetilde{\Omega}$. Then

$$K_{\widetilde{\Omega}}(\widetilde{w}) = (K_\Omega(w))^m$$

and by [13] (see also [9])

$$G_{\widetilde{\Omega},\widetilde{w}}(z^1, \dots, z^m) = \max_{j=1,\dots,m} G(z^j),$$

therefore

$$\lambda_{2nm}(\{G_{\widetilde{\Omega},\widetilde{w}} < -a\}) = (\lambda_{2n}(\{G < -a\})^m.$$

It follows from the previous part that

$$(K_\Omega(w))^m \geq \frac{c(nm,a)}{(\lambda_{2n}(\{G < -a\}))^m}$$

and it is enough to use the fact that

$$\lim_{m\to\infty} c(nm,a)^{1/m} = e^{-2na}.$$

□

Theorem 2 follows immediately from Theorem 1 and the following result by approximation.

Proposition 3. *Assume that Ω is a bounded, smooth, strongly convex domain in $\mathbb{C}^n$. Then for any $w \in \Omega$*

$$\lim_{a\to\infty} e^{2na}\lambda_{2n}(\{G_{\Omega,w} < -a\}) = \lambda_{2n}(I_\Omega(w)). \tag{8}$$

Proof. Denote $I := I_\Omega(w)$, $G := G_{\Omega,w}$, we may assume that $w = 0$. By the results of Lempert [15] there exists a diffeomorphism $\Phi : \bar{I} \to \bar{\Omega}$ such that for $v \in \partial I$ the mapping $\Delta \ni \zeta \mapsto \Phi(\zeta v)$ is a geodesic in Ω, that is

$$G(\Phi(\zeta v)) = \log|\zeta|. \tag{9}$$

(Φ can be treated as an exponential map for the Kobayashi distance.) We also have

$$\Phi(\zeta v) = \zeta v + O(|\zeta|^2).$$

By (9)

$$\{G < -a\} = \Phi(e^{-a}\mathrm{int}\, I)$$

and therefore

$$\lambda_{2n}(\{G < -a\}) = \int_{e^{-a}I} \mathrm{Jac}\,\Phi\, d\lambda_{2n}.$$

Since $\Phi'(0)$ is the identity, we obtain (8). □

3 Applications to the Bourgain-Milman Inequality

Assume that L is a convex symmetric body in $\mathbb{R}^n$. In view of Theorem 2, in order to prove Nazarov's lower bound (6) it is enough to show the estimate

$$\lambda_{2n}(I_{T_L}(0)) \le \left(\frac{4}{\pi}\right)^{2n} (\lambda_n(L))^2. \tag{10}$$

But this follows immediately from the following:

Proposition 4. $I_{T_L}(0) \subset \dfrac{4}{\pi}(\bar{L} + i\bar{L})$.

Proof. We will use an idea of Nazarov [17] here. Let Φ be a conformal mapping from the strip $\{|\mathrm{Re}\,\zeta| < 1\}$ to Δ with $\Phi(0) = 0$, so that $|\Phi'(0)| = \pi/4$. For $u \in L'$ we can then define $F \in \mathcal{O}(T_L, \Delta)$ by $F(z) = \Phi(z \cdot u)$. For $\varphi \in \mathcal{O}(\Delta, T_L)$ with $\varphi(0) = 0$ by the Schwarz lemma we have $|(F \circ \varphi)'(0)| \le 1$. Therefore $|\varphi'(0) \cdot u| \le 4/\pi$ and $\frac{\pi}{4} I_{T_L}(0) \subset L_{\mathbb{C}}$, where

$$L_{\mathbb{C}} = \{z \in \mathbb{C}^n : |z \cdot u| \le 1 \text{ for all } u \in L'\} \subset \bar{L} + i\bar{L}.$$

□

It will be convenient to use the notation $J_L := \frac{\pi}{4} I_{T_L}(0)$, so that by the proof of Proposition 4

$$J_L \subset L_{\mathbb{C}} \subset \bar{L} + i\bar{L}. \tag{11}$$

We thus have $\lambda_{2n}(J_L) \le (\lambda_n(L))^2$ but we conjecture that

$$\lambda_{2n}(J_L) \le \left(\frac{\pi}{4}\right)^n (\lambda_n(L))^2. \tag{12}$$

Note that $J_{[-1,1]^n} = \Delta^n$, so that we have equality for cubes. The inequality (12) would give the optimal lower bound for the Bergman kernel in tube domains (7).

We first give an example that (11) cannot give us (12):

Example. Let $L = \{x_1^2 + x_2^2 < 1\}$ be the unit disc in $\mathbb{R}^2$. One can then show that $L_{\mathbb{C}} = \{|z|^2 \le 1 + (x_1 y_2 - x_2 y_1)^2\}$ and

$$\lambda_4(L_{\mathbb{C}}) = \frac{2\pi^2}{3} > \frac{\pi^4}{16} = \left(\frac{\pi}{4}\right)^2 (\lambda_2(L))^2.$$

4 Lempert's Theory in Tube Domains

Our goal is to approach (12) using Lempert's theory. First assume that Ω is bounded, smooth, strongly convex domain in $\mathbb{C}^n$. Then for any $z, w \in \Omega$, $z \neq w$, there exists unique extremal disc $\varphi \in \mathcal{O}(\Delta, \Omega) \cap C^\infty(\bar{\Delta}, \bar{\Omega})$ such that $\varphi(0) = w$, $\varphi(\xi) = z$ for some ξ with $0 < \xi < 1$, and

$$G_{\Omega,w}(\varphi(\zeta)) = \log|\zeta|, \quad \zeta \in \Delta.$$

Lempert [15] showed in particular the following characterization of extremal discs: a disc $\varphi \in \mathcal{O}(\Delta, \Omega) \cap C(\bar{\Delta}, \bar{\Omega})$ is extremal if and only if $\varphi(\partial\Delta) \subset \partial\Omega$ and there exists $h \in \mathcal{O}(\Delta, \mathbb{C}^n) \cap C(\bar{\Delta}, \mathbb{C}^n)$ such that the vector $e^{it}\overline{h(e^{it})}$ is outer normal to $\partial\Omega$ at $\varphi(e^{it})$ for every $t \in \mathbb{R}$.

Lempert [15] also proved that for every extremal disc φ in Ω there exists a left-inverse $F \in \mathcal{O}(\Omega, \Delta)$ (that is $F(\varphi(\zeta)) = \zeta$ for $\zeta \in \Delta$). It solves the equation

$$(z - \varphi(F(z))) \cdot h(F(z)) = 0, \quad z \in \Omega. \tag{13}$$

Now assume that L is a smooth, strongly convex body in $\mathbb{R}^n$. Although T_L is neither bounded nor strongly convex, we may nevertheless try to apply Lempert's condition for extremal discs (the details have been worked out by Zając [20]). First note that $h \in \mathcal{O}(\Delta, \mathbb{C}^n) \cap C(\bar{\Delta}, \mathbb{C}^n)$ in our case must be such that $e^{it}\overline{h(e^{it})}$ is an outer normal to T_L and therefore its imaginary part vanishes:

$$\mathrm{Im}\,(e^{-it}h(e^{it})) = 0, \quad t \in \mathbb{R}. \tag{14}$$

It follows that h must be of a very special form:

Lemma 5 ([20]). *If $h \in \mathcal{O}(\Delta) \cap C(\bar{\Delta})$ satisfies (14) then $h(\zeta) = a + b\zeta + \bar{a}\zeta^2$ for some $a \in \mathbb{C}$ and $b \in \mathbb{R}$.*

Proof. Set $a := h(0)$. Then for $\zeta \in \partial\Delta$

$$0 = \mathrm{Im}\left(\frac{h(\zeta)}{\zeta}\right) = \mathrm{Im}\left(\frac{h(\zeta) - a}{\zeta} - \bar{a}\zeta\right)$$

and therefore

$$\frac{h(\zeta) - a}{\zeta} - \bar{a}\zeta = b \in \mathbb{R}, \quad \zeta \in \bar{\Delta}.$$

□

We thus see that in our case h must be of the form

$$h(\zeta) = w + \zeta b + \zeta^2\bar{w}, \quad \zeta \in \bar{\Delta},$$

for some $w \in \mathbb{C}^n$ and $b \in \mathbb{R}^n$. Take the extremal disc φ for T_L associated with h. Since $e^{it}\overline{h(e^{it})}$ is an outer normal to T_L at $\varphi(e^{it})$ and its imaginary part vanishes, it follows that its real part is an outer normal to L at $\mathrm{Re}\,(\varphi(e^{it}))$. Therefore

$$\mathrm{Re}\,\varphi(e^{it}) = \nu^{-1}\left(\frac{b + \mathrm{Re}\,(e^{-it}w)}{|b + \mathrm{Re}\,(e^{-it}w)|}\right), \tag{15}$$

where

$$\nu : \partial L \to \mathbb{S}^{n-1}$$

is the Gauss map.

For $\varphi \in \mathcal{O}(\Delta) \cap C(\bar{\Delta})$ we can recover the values of φ in Δ from the values of $\mathrm{Re}\,\varphi$ on $\partial\Delta$ using the Schwarz formula:

$$\varphi(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+\zeta}{e^{it}-\zeta}\mathrm{Re}\,\varphi(e^{it})\,dt + i\,\mathrm{Im}\,\varphi(0), \quad \zeta \in \Delta.$$

Therefore extremal discs satisfying (15) are given by

$$\varphi(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+\zeta}{e^{it}-\zeta}\,\nu^{-1}\left(\frac{b + \mathrm{Re}\,(e^{-it}w)}{|b + \mathrm{Re}\,(e^{-it}w)|}\right) dt + i\,\mathrm{Im}\,\varphi(0), \quad \zeta \in \Delta.$$

We now assume that L is in addition symmetric and then consider the case when $b = 0$ and $\mathrm{Im}\,\varphi(0) = 0$:

$$\varphi(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+\zeta}{e^{it}-\zeta}\,\nu^{-1}\left(\frac{\mathrm{Re}\,(e^{-it}w)}{|\mathrm{Re}\,(e^{-it}w)|}\right) dt. \tag{16}$$

Since L is symmetric the function $B(t)$ under the integral in (16) satisfies $B(t+\pi) = -B(t)$. We thus have $\varphi(0) = 0$ and one can show (see [20] for details) that all geodesics of T_L passing through the origin are given by (16). They are bounded and smooth up to the boundary if $\mathrm{Re}\,w$ and $\mathrm{Im}\,w$ are linearly independent in $\mathbb{R}^n$. If $\mathrm{Re}\,w$ and $\mathrm{Im}\,w$ are linearly dependent (and $w \neq 0$) then (16) gives special extremal discs of the form

$$\varphi(\zeta) = \Phi^{-1}(\zeta)\,x, \quad x \in \partial L,$$

where Φ is as in the proof Proposition 4. Left-inverses to these φ are then given by $F(z) = \Phi(z \cdot u)$ for unique $u \in \partial L'$ with $x \cdot u = 1$.

For geodesics (16) we have

$$\varphi'(0) = \frac{1}{\pi}\int_0^{2\pi} e^{it}\,\nu^{-1}\left(\frac{\mathrm{Re}\,(e^{it}w)}{|\mathrm{Re}\,(e^{it}w)|}\right) dt. \tag{17}$$

These vectors parametrize the boundary of the Kobayashi indicatrix $I_L(0)$. If $F \in \mathcal{O}(\Omega, \Delta)$ is the left-inverse of φ satisfying (13) we get, since $h'(0) = 0$,

$$F'(0) = \frac{w}{\varphi'(0) \cdot w}.$$

Therefore

$$J_L = \{z \in \mathbb{C}^n : |z \cdot w| \leq |\Psi(w)| \text{ for all } w \in (\mathbb{C}^n)_*\}, \tag{18}$$

where

$$\Psi(w) = \frac{1}{4} \int_0^{2\pi} e^{it} w \cdot \nu^{-1} \left(\frac{\mathrm{Re}\,(e^{it} w)}{|\mathrm{Re}\,(e^{it} w)|} \right) dt.$$

Both (17) and (18) give a description of the set J_L. It would be interesting to try to use it to prove (12). We can at least show this for a ball:

Example. Let $B = \{|x| < 1\}$ be the unit ball in $\mathbb{R}^n$. For $w \in (\mathbb{C}^n)_*$ we have

$$\mathrm{Im}\,\Psi(w) = \frac{1}{4} \int_0^{2\pi} \frac{\mathrm{Im}\,(e^{it} w) \cdot \mathrm{Re}\,(e^{it} w)}{|\mathrm{Re}\,(e^{it} w)|} dt = -\frac{1}{4} \int_0^{2\pi} \frac{d}{dt} |\mathrm{Re}\,(e^{it} w)| dt = 0$$

and thus

$$\Psi(w) = \frac{1}{4} \int_0^{2\pi} |\mathrm{Re}\,(e^{it} w)| dt \leq \frac{\pi}{\sqrt{8}} |w|.$$

By (18) J_B is contained in a ball with radius $\pi/\sqrt{8}$ in $\mathbb{C}^n$. Therefore

$$\lambda_{2n}(J_B) \leq \frac{\pi^{3n}}{8^n n!}.$$

On the other hand,

$$\lambda_n(B) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)},$$

and we see that (12) holds for B if $n \geq 3$. To show this also for $n = 2$ we have to use in addition Proposition 4: $J_B \subset (\bar{B} + i\bar{B}) \cap (r_0 \mathbb{B})$, where $r_0 = \pi/\sqrt{8}$. With $\rho_0 = \sqrt{r_0^2 - 1}$ we will get

$$\lambda_4(J_B) \leq \pi^2 \rho_0^2 + \pi^2 \int_{\rho_0}^1 \rho(r_0^2 - \rho^2) d\rho = \frac{\pi^6}{256} + \frac{\pi^4}{16} - \frac{\pi^2}{2} < \frac{\pi^4}{16} = \left(\frac{\pi}{4}\right)^2 (\lambda_2(B))^2.$$

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