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Some applications of the Ohsawa–Takegoshi extension theorem[☆]

Zbigniew Błocki
Institute of Mathematics, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland

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Abstract

We sketch a proof of the Ohsawa–Takegoshi extension theorem (due to Berndtsson) and then present some applications of this result: optimal lower bound for the Bergman kernel, relation to the Suita conjecture, and the Demailly approximation.

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0. Introduction

This is an expanded version of the guest lecture, given on May 2, 2007, as a part of the course *Several Complex Variables* of professor Ngaiming Mok at the University of Hong Kong. The aim is to discuss the Ohsawa–Takegoshi extension theorem (in its original form for domains in $\mathbb{C}^n$) and some applications. We sketch a proof due to Berndtsson (which is more in the spirit of Hörmander's book [H], rather than more complicated methods of abstract Kähler manifolds used in [OT]) and then present some consequences of the

E-mail addresses: zbigniew.blocki@im.uj.edu.pl, umblocki@cyf-kr.edu.pl.

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extension theorem for the Bergman kernel. The Suita conjecture, although formally not a part of Several Complex Variables, because of its close relation with the extension theorem, is discussed in detail. Finally, the basic results of the approximation of plurisubharmonic functions introduced by Demailly are presented. They follow quite easily from the extension theorem, and since they also imply in a simple way the Siu theorem on analyticity of level sets of Lelong numbers of a plurisubharmonic function, this again demonstrates the power of the Ohsawa–Takegoshi theorem.

We limit our interest only to domains in $\mathbb{C}^n$ and holomorphic functions. There are many generalizations of the extension theorem to manifolds, sections of vector bundles, etc., but this is beyond the scope of this presentation.

One of the main materials used when preparing these notes were Ż. Dinew master thesis [Di1]. Some of the results presented below can also be found in Ohsawa's booklet [O5].

1. The Ohsawa–Takegoshi extension theorem

The following result was proved in [OT].

Theorem 1.1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and φ an arbitrary plurisubharmonic function in Ω. Assume that H is a complex (linear) subspace of $\mathbb{C}^n$ and denote $\Omega' := \Omega \cap H$. Then for every holomorphic function f on Ω' there exists a holomorphic F in Ω such that $F = f$ on Ω' and*

$$\int_\Omega |F|^2 e^{-\varphi}\, d\lambda \leqslant C_\Omega \int_{\Omega'} |f|^2 e^{-\varphi}\, d\lambda',$$

where C_Ω is a constant depending only on n and on an upper bound for the diameter of Ω ($d\lambda$ denotes the Lebesgue measure on $\mathbb{C}^n$ and $d\lambda'$ the Lebesgue measure on H).

Before sketching the proof of this theorem let us briefly discuss the L^2-theory of the $\bar\partial$-equation. For a given (0,1)-form α with $\bar\partial\alpha = 0$ (which is a necessary condition) one is interested in solving

$$\bar\partial u = \alpha \tag{1.1}$$

with an L^2-estimate. Such solutions are very useful in constructing new holomorphic functions because $\bar\partial v = 0$ implies that v is holomorphic. The most classical result in this direction is due to Hörmander [H] (see also [D1]).

Hörmander's estimate. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and φ a C^2 strongly plurisubharmonic function in Ω. Then for every $\alpha \in L^2_{loc,(0,1)}(\Omega)$ with $\bar\partial\alpha = 0$ there exists $u \in L^2_{loc}(\Omega)$ solving (1.1) and such that*

$$\int_\Omega |u|^2 e^{-\varphi}\, d\lambda \leqslant \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{-\varphi}\, d\lambda.$$

Here

$$|\alpha|^2_{i\partial\overline{\partial}\varphi} = \sum_{j,k} \varphi^{j\bar{k}}\overline{\alpha}_j \alpha_k$$

(where $\alpha = \sum \alpha_j d\overline{z}_j$ and $(\varphi^{k\bar{j}})$ is the inverse of $(\varphi_{j\bar{k}}) = (\partial^2\varphi/\partial z_j \partial\overline{z}_k))$ denotes the length of the form α w.r.t. the Kähler metric $i\partial\overline{\partial}\varphi$. In fact, the estimate makes sense also for an arbitrary plurisubharmonic φ: for h with $i\overline{\alpha} \wedge \alpha \leqslant hi\partial\overline{\partial}\varphi$ we can find u with

$$\int_\Omega |u|^2 \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda \leqslant \int_\Omega h\mathrm{e}^{-\varphi}\,\mathrm{d}\lambda$$

(see [Bł1] or [Bł3]).

The Hörmander estimate has been one of the most useful results in the study of holomorphic functions of several variables. Another interesting and useful estimate is due to Donnelly and Fefferman [DoF] (this paper in fact influenced [OT]).

Donnelly–Fefferman's estimate. *Let Ω, φ, and α be as in Hörmander's estimate. Assume in addition that ψ is plurisubharmonic in Ω and such that $i\partial\psi \wedge \overline{\partial}\psi \leqslant i\partial\overline{\partial}\psi$ (which is equivalent to the fact that $\psi = -\log(-v)$ for some negative plurisubharmonic v). Then there exists $u \in L^2_{loc}(\Omega)$ solving* (1.1) *and such that*

$$\int_\Omega |u|^2 \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda \leqslant 4\int_\Omega |\alpha|^2_{i\partial\overline{\partial}\psi} \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda.$$

Berndtsson [Be3] showed that Donnelly–Fefferman's estimate is a formal consequence of Hörmander's estimate (see also [Bł2]), which is not the case with the Ohsawa–Takegoshi theorem.

We will now sketch a recent proof of Theorem 1.1 due to Berndtsson.

Proof of Theorem 1.1. [Sketch] We follow [Be4] (see also [Be2]). Without loss of generality we may assume that $H = \{z_1 = 0\}$ and $\Omega \subset \{|z_1| < 1\}$. By approximating Ω from inside and φ from above we may assume that Ω is a strongly pseudoconvex domain with smooth boundary, φ is smooth up to the boundary, and f is defined in a neighborhood of $\overline{\Omega'}$ in H. Then it follows that f extends to some holomorphic function in Ω (we may use Hörmander's estimate with $\alpha = \overline{\partial}(\chi(z_1)f(z'))$, $\chi = 1$ near 0 but with support sufficiently close to 0, $\varphi = 2\log|z_1|$ will ensure that $u = 0$ on H).

Let $F \in H^2(\Omega, \mathrm{e}^{-\varphi}) := \mathcal{O}(\Omega) \cap L^2(\Omega, \mathrm{e}^{-\varphi})$ be the function satisfying $F = f$ on H with minimal norm in $L^2(\Omega, \mathrm{e}^{-\varphi})$. Then F is perpendicular to functions from $H^2(\Omega, \mathrm{e}^{-\varphi})$ vanishing on H, and it is thus perpendicular to the space $z_1 H^2(\Omega, \mathrm{e}^{-\varphi})$.This means that $\overline{z}_1 F$ is perpendicular to $H^2(\Omega, \mathrm{e}^{-\varphi})$. Since $(H^2(\Omega, \mathrm{e}^{-\varphi}))^\perp = (\ker\overline{\partial})^\perp$ is equal to the range of $\overline{\partial}^*$, we have $\overline{\partial}^*\alpha = \overline{z}_1 F$ for some $\alpha \in L^2_{(0,1)}(\Omega, \mathrm{e}^{-\varphi})$. Choose such α with the minimal norm. Then α is perpendicular to $\ker\overline{\partial}^*$, and thus $\overline{\partial}\alpha = 0$. One can also show that the tangential component of α vanishes on the boundary (or in other words α satisfies $\overline{\partial}$-Neumann boundary condition), that is

$$\sum_j \alpha_j \rho_j = 0 \quad \text{on}\,\partial\Omega, \tag{1.2}$$

where ρ is a defining function for Ω. We have

$$\begin{aligned}\int_\Omega |F|^2 e^{-\varphi}\, d\lambda &= \langle F/z_1, \bar\partial^*\alpha\rangle_{e^{-\varphi}} = \langle \bar\partial(F/z_1), \alpha\rangle_{e^{-\varphi}} = \langle F\bar\partial(1/z_1), \alpha\rangle_{e^{-\varphi}}\\ &= \pi\int_{\Omega'} f\bar\alpha_1 e^{-\varphi}\, d\lambda' \leqslant \pi\left(\int_{\Omega'} |f|^2 e^{-\varphi}\, d\lambda'\right)^{1/2}\left(\int_{\Omega'} |\alpha_1|^2 e^{-\varphi}\, d\lambda'\right)^{1/2}.\end{aligned} \tag{1.3}$$

It is thus enough to estimate $\int_{\Omega'} |\alpha_1|^2 e^{-\varphi}\, d\lambda'$. We will use the Bochner–Kodaira technique (terminology of Siu [S2], see [Be2] for details). One may compute that

$$\begin{aligned}\sum(\alpha_j\bar\alpha_k e^{-\varphi})_{j\bar k} = (&-2\,\mathrm{Re}\,(\bar\partial\bar\partial^*\alpha\cdot\alpha) + |\bar\partial^*\alpha|^2 + \sum|\alpha_{j,\bar k}|^2 - |\bar\partial\alpha|^2\\ &+ \sum \varphi_{j\bar k}\alpha_j\bar\alpha_k)e^{-\varphi}.\end{aligned}$$

Integrating by parts and computing further one can show that for any (sufficiently regular) form α satisfying (1.2) and a function w

$$\begin{aligned}&\int_\Omega \sum w_{j\bar k}\alpha_j\bar\alpha_k e^{-\varphi}\, d\lambda - \int_{\partial\Omega}\sum \rho_{j\bar k}\alpha_j\bar\alpha_k e^{-\varphi} w\frac{d\sigma}{|\partial\rho|}\\ &\quad = \int_\Omega (-2\,\mathrm{Re}\,(\bar\partial\bar\partial^*\alpha\cdot\alpha) + |\bar\partial^*\alpha|^2 + \sum|\alpha_{j,\bar k}|^2 - |\bar\partial\alpha|^2 + \sum\varphi_{j\bar k}\alpha_j\bar\alpha_k)e^{-\varphi} w\, d\lambda.\end{aligned}$$

In our case we have $\bar\partial\alpha = 0$, $\bar\partial^*\alpha = \bar z_1 F$, and if we take negative w depending only on z_1, then

$$\int_\Omega w_{1\bar 1}|\alpha_1|^2 e^{-\varphi}\, d\lambda \leqslant -2\,\mathrm{Re}\int_\Omega F\bar\alpha_1 e^{-\varphi} w\, d\lambda \tag{1.4}$$

(since we may choose plurisubharmonic ρ). Set

$$w := 2\log|z_1| + |z_1|^{2\delta} - 1,$$

where $0<\delta<1$. Then $w_{1\bar 1} = \pi\delta'_0 + \delta^2|z_1|^{2\delta-2}$ and for $t>0$

$$\begin{aligned}&\pi\int_{\Omega'} |\alpha_1|^2 e^{-\varphi}\, d\lambda' + \delta^2\int_\Omega |\alpha_1|^2|z_1|^{2\delta-2}e^{-\varphi}\, d\lambda\\ &\quad\leqslant t\int_\Omega |F|^2 e^{-\varphi}\, d\lambda + \frac{1}{t}\int_\Omega |\alpha_1|^2 w^2 e^{-\varphi}\, d\lambda.\end{aligned}$$

Choosing t with $w^2 \leqslant \delta^2 t|z_1|^{2\delta-2}$ in $\{|z_1|\leqslant 1\}$ and combining this with (1.4) we arrive at

$$\int_\Omega |F|^2 e^{-\varphi}\, d\lambda \leqslant t\pi\int_{\Omega'} |f|^2 e^{-\varphi}\, d\lambda'. \qquad \square$$

2. The Bergman kernel

In this section we will present some applications of Theorem 1.1 related to the Bergman kernel (on the diagonal)

$$K_\Omega := \sup\left\{ |f|^2 : f \in \mathcal{O}(\Omega), \int_\Omega |f|^2 \, d\lambda \leqslant 1 \right\}.$$

For a ball we have

$$K_{B(z_0,r)}(z) = \frac{n! r^2}{\pi^n (r^2 - |z - z_0|^2)^{n+1}}.$$

Note that from Theorem 1.1 we immediately get

$$K_{\Omega'} \leqslant C_\Omega K_\Omega \quad \text{on } \Omega'. \tag{2.1}$$

The original motivation (see [O1]) behind [OT] was the following estimate.

Theorem 2.1. *Assume that Ω is a bounded pseudoconvex domain with C^2 boundary. Then*

$$K_\Omega \geqslant \frac{1}{C \delta_\Omega^2}, \tag{2.2}$$

where C is a constant depending on Ω and $\delta_\Omega(z)$ is the euclidean distance from $z \in \Omega$ to $\partial\Omega$.

Proof. It follows almost immediately from Theorem 1.1 For let $r > 0$ be such that for any $w \in \partial\Omega$ there exists $w^* \in \mathbb{C}^n \setminus \overline{\Omega}$ such that $\overline{\Omega} \cap \overline{B}(w^*, r) = \{w\}$. If $z \in \Omega$, $w \in \partial\Omega$ is such that $\delta_\Omega(z) = |z - w|$, and w^* is as above then z, w, and w^* lie on the same line (normal to $\partial\Omega$ at w). For the corresponding complex line H and $\Omega' = \Omega \cap H$ we obtain

$$K_\Omega(z) \geqslant \frac{1}{C_\Omega} K_{\Omega'}(z) \geqslant \frac{1}{C_\Omega} K_{\mathbb{C}\setminus\overline{\Delta}(0,r)}(r + |z - w|) = \frac{r^2}{\pi C_\Omega \delta_\Omega(z)^2 (2r + \delta_\Omega(z))^2}. \qquad \square$$

The exponent 2 in (2.2) is optimal (for example it cannot be improved for a domain whose boundary near the origin is given by $|z_1 - 1| = 0$). Previously a weaker form of (2.2) was proved by Pflug [P] using Hörmander's estimate (with arbitrary exponent lower than 2).

A domain is called hyperconvex if it admits a bounded plurisubharmonic exhaustion. Of course every hyperconvex domain is pseudoconvex. In dimension 1 hyperconvexity is equivalent to regularity. In higher dimensions it was proved by Kerzman and Rosay [KR] that hyperconvexity is a local property of the boundary and by Demailly [D2] that any pseudoconvex domain with Lipschitz boundary is hyperconvex. (That any bounded pseudoconvex domain with C^2 boundary is hyperconvex follows from an earlier result of Diederich and Fornæss [DF].)

Theorem 1.1 was used in [O2] to prove the following result.

Theorem 2.2. *For a bounded hyperconvex domain Ω in $\mathbb{C}^n$ one has*

$$\lim_{z \to \partial\Omega} K_\Omega(z) = \infty.$$

The proof in [O2] consists of two steps: first a quantitative (in terms of potential theory) lower bound for the Bergman kernel was shown in dimension 1 (using Hörmander's estimate), and then Theorem 2.2 followed easily from the extension theorem. In fact, one can prove Theorem 2.2 using the complex Monge–Ampére operator instead of Theorem 1.1 (see [Bł1] or [Bł4]).

The extension theorem is also used in the following interesting result of Pflug and Zwonek [PZ].

Theorem 2.3. *For a bounded pseudoconvex domain Ω in $\mathbb{C}^n$ the following are equivalent*:

(i) *Ω is an L^2-domain of holomorphy* (*that is Ω is a domain of existence of a function from $H^2(\Omega)$*);
(ii) *$\partial\Omega$ has no pluripolar part* (*that is if U is open then $U \cap \partial\Omega$ is either empty or non-pluripolar*);
(iii) $\limsup_{z\to w} K_\Omega(z) = \infty$, $w \in \partial\Omega$.

3. The Suita conjecture

As noticed by Ohsawa [O3], Theorem 2.2 is closely related to the following conjecture of Suita [Su]: for a bounded domain D in $\mathbb{C}$ one has

$$c_D^2 \leqslant \pi K_D, \tag{3.1}$$

where

$$c_D(\zeta) = \exp \lim_{\eta\to\zeta}(G_D(\eta,\zeta) - \log|\eta-\zeta|), \quad \zeta \in D,$$

is the logarithmic capacity of $\mathbb{C}\backslash D$ w.r.t. ζ (G_D denotes the Green function of D with negative sign). It is easy to show that equality holds in (3.1) if D is simply connected, and Suita [Su] showed strict inequality in (3.1) when D is an annulus (and thus also when it is any smooth doubly connected domain). By approximation, it is enough to prove (3.1) for bounded m-connected domains with smooth boundary.

It is perhaps interesting that so far only the methods of several complex variables proved successful in this kind of problem for an arbitrary domain. The relation of the Suita conjecture to the extension theorem stems from the fact that (3.1) is equivalent to the following statement: for every $\zeta \in D$ there exists $f \in \mathcal{O}(\Omega)$ such that $f(\zeta) = 1$ and

$$\int_D |f|^2 \, d\lambda \leqslant \frac{\pi}{c_D^2(\zeta)}.$$

Ohsawa [O3], using the methods of the proof of the extension theorem, proved the estimate

$$c_D^2 \leqslant C K_D \tag{3.2}$$

with $C = 750\pi$. This constant has been improved in [Bł5] to 2π (the proof in [Bł5] does not use the L^2 theory, only an estimate of Berndtsson [Be1] for a solution of a one-dimensional $\bar\partial$-Neumann problem).

The Suita conjecture may thus be reformulated as follows: the optimal constant in the Ohsawa estimate (3.2) is π. It is however not just about an optimal constant. For let

$$\psi(\zeta) := \lim_{\eta\to\zeta}(G_D(\eta,\zeta) - \log|\eta-\zeta|),$$

so that $c_D = \mathrm{e}^{\psi}$. From the Schiffer formula saying that the expression

$$\frac{2}{\pi}\frac{\partial^2 G_D}{\partial z\partial\overline{w}}$$

is equal to the Bergman kernel off the diagonal (see e.g. [B]) one can easily show that $K_D = \psi_{\zeta\bar\zeta}/\pi$ (this was proved originally in [Su] with a more complicated argument). Therefore (3.1) reads

$$-\frac{\psi_{\zeta\bar\zeta}}{\mathrm{e}^{2\psi}} \leqslant -1, \tag{3.1$'$}$$

and the left-hand side is precisely the curvature of the metric $\mathrm{e}^{\psi}|\mathrm{d}\zeta|$. One can show that if D has a smooth boundary then the left-hand side of (3.1$'$) is smooth up to the boundary and we have equality in (3.1$'$) on the boundary. The Suita conjecture thus predicts that the curvature satisfies the maximum principle in D (globally). It is perhaps worth noticing that this property does not hold for the metrics $\psi_{\zeta\bar\zeta}|\mathrm{d}\zeta|^2$ and $\log(\psi_{\zeta\bar\zeta})_{\zeta\bar\zeta}|\mathrm{d}\zeta|^2$ (the latter is precisely the Bergman metric) already in an annulus (this can be deduced from the computations in [Su]).

The estimate (3.2) does not follow from the statement of Theorem 1.1. Ohsawa [O4] generalized Theorem 1.1 so that it includes (3.2). The following result is a variation of this and can be found in [Di2].

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ such that $\Omega \subset D\times\mathbb{C}^{n-1}$ for some domain D in $\mathbb{C}$ containing the origin. Let $H := \{z_1 = 0\}$ and $\Omega' = \Omega\cap H$. Then for any $f \in \mathcal{O}(\Omega')$ and $\varphi \in PSH(\Omega)$ there exists $F \in \mathcal{O}(\Omega)$ with $F = f$ on Ω' and*

$$\int_\Omega |F|^2\mathrm{e}^{-\varphi}\,\mathrm{d}\lambda \leqslant \frac{4\pi}{c_D(0)^2}\int_{\Omega'}|f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}\lambda'.$$

The right formulation of the Suita conjecture in several variables seems to be that the constant in Theorem 3.1 can be improved to $\pi/c_D(0)^2$.

4. The Demailly approximation

So far we have used Theorem 1.1 only with $\varphi \equiv 0$. The fact that the weight may be an arbitrary plurisubharmonic function was used by Demailly [D3] to introduce a new type of regularization of plurisubharmonic functions: by smooth plurisubharmonic functions with analytic singularities (that is functions that locally can be written in the form $\log(|f_1|^2+\cdots+|f_k|^2)+u$,where $f_1,\ldots,f_k$ are holomorphic and u is C^∞ smooth) which have very similar singularities to the initial function. The Demailly approximation turned out to be an important tool in complex geometry,see e.g. [D3,DPS,DP] or [Po]. Demailly [D3] presented also a simple proof of the Siu theorem on analyticity of level sets of Lelong numbers of plurisubharmonic functions ([S1],see also [H]). As we will see below,the Demailly approximation shows that the Siu theorem follows rather easily from Theorem 1.1 applied when H is just a point (note that in this case Theorem 1.1 is trivial for $\varphi \equiv 0$,but it is no longer true for Theorem 3.1).

Recall that the Lelong number of $\varphi \in PSH(\Omega)$ at $z_0 \in \Omega$ is defined by

$$\nu_\varphi(z_0) = \varliminf_{z \to z_0} \frac{\varphi(z)}{\log|z - z_0|} = \lim_{r \to 0^+} \frac{\varphi^r(z_0)}{\log r},$$

where for $r > 0$ we use the notation

$$\varphi^r(z) := \max_{\overline{B}(z,r)} \varphi, \quad z \in \Omega_r := \{\delta_\Omega > r\}.$$

One can show that φ^r is a plurisubharmonic continuous function in Ω_r, decreasing to φ as r decreases to 0. Now we are in position to prove a result from [D3].

Theorem 4.1. *For a plurisubharmonic function φ in a bounded pseudoconvex domain Ω in $\mathbb{C}^n$ and $m = 1, 2...$ set*

$$\varphi_m := \frac{1}{2m} \log K_{\Omega, \mathrm{e}^{-2m\varphi}} = \frac{1}{2m} \log \sup \left\{ |f|^2 : f \in \mathcal{O}(\Omega), \int_\Omega |f|^2 \mathrm{e}^{-2m\varphi} \leqslant 1 \right\}.$$

Then there exist $C_1, C_2 > 0$ depending only on Ω such that

$$\varphi - \frac{C_1}{m} \leqslant \varphi_m \leqslant \varphi^r + \frac{1}{m} \log \frac{C_2}{r^n} \quad \text{in } \Omega_r. \tag{4.1}$$

In particular, $\varphi_m \to \varphi$ pointwise and in $L^1_{loc}(\Omega)$. Moreover,

$$\nu_\varphi - \frac{n}{m} \leqslant \nu_{\varphi_m} \leqslant \nu_\varphi \quad \text{in } \Omega. \tag{4.2}$$

Proof. First note that (4.2) is an easy consequence of (4.1): by the first inequality in (4.1) we get $\nu_{\varphi_m} \leqslant \nu_{\varphi - C_1/m} = \nu_\varphi$, and by the second one

$$\varphi_m^r \leqslant \varphi^{2r} + \frac{1}{m} \log \frac{C_2}{r^n},$$

thus $\nu_\varphi - n/m \leqslant \nu_{\varphi_m}$.

By Theorem 1.1 for every $z \in \Omega$ there exists $f \in \mathcal{O}(\Omega)$ with $f(z) \neq 0$ and

$$\int_\Omega |f|^2 \mathrm{e}^{-2m\varphi} \, \mathrm{d}\lambda \leqslant C_\Omega |f(z)|^2 \mathrm{e}^{-2m\varphi(z)}.$$

We may choose f so that the right-hand side is equal to 1. Then

$$\varphi_m(z) \geqslant \frac{1}{m} \log|f(z)| = \varphi(z) - \frac{1}{2m} \log C_\Omega$$

and we get the first inequality in (4.1).

To get the second one we observe that for any holomorphic f the function $|f|^2$ is in particular subharmonic and thus for $z \in \Omega_r$

$$|f(z)|^2 \leqslant \frac{1}{\lambda(B(z,r))} \int_{B(z,r)} |f|^2 \, d\lambda \leqslant \frac{n!}{\pi^n r^{2n}} e^{2m\varphi^r(z)} \int_{B(z,r)} |f|^2 e^{-2m\varphi} \, d\lambda.$$

Taking the logarithm and multiplying by $1/(2m)$ we will easily get the second inequality in (4.1). $\square$

By (4.2) for any real c we have

$$\{v_\varphi \geqslant c\} = \bigcap_m \left\{ v_{\varphi_m} \geqslant c - \frac{n}{m} \right\}. \tag{4.3}$$

If $\{\sigma_j\}$ is an orthonormal basis in $H^2(\Omega, e^{-2m\varphi})$ then

$$K_{\Omega, e^{-2m\varphi}} = \sum_j |\sigma_j|^2 \tag{4.4}$$

and one can show that

$$\left\{ v_{\varphi_m} \geqslant c - \frac{n}{m} \right\} = \bigcap_{|\alpha| < mc - n} \bigcap_j \{\partial^\alpha \sigma_j = 0\}.$$

Therefore (4.3) is an analytic subset of Ω, which gives the Siu theorem [S1].

Theorem 4.2. *For any plurisubharmonic function φ and a real number c the set $\{v_\varphi \geqslant c\}$ is analytic.*

The following sub-additivity property was proved in [DPS]. It also relies on the extension theorem, here, however, we will be using it for the diagonal of $\Omega \times \Omega$.

Theorem 4.3. *With the notation of* Theorem 4.1 *there exists $C_3 > 0$, depending only on Ω, such that*

$$(m_1 + m_2)\varphi_{m_1+m_2} \leqslant C_3 + m_1 \varphi_{m_1} + m_2 \varphi_{m_2}.$$

Proof. Take $f \in H^2(\Omega, e^{-2(m_1+m_2)\varphi})$ with norm $\leqslant 1$. If we embed Ω in $\Omega \times \Omega$ as the diagonal then by Theorem 1.1 there exists F holomorphic in $\Omega \times \Omega$ such that $F(z,z) = f(z)$, $z \in \Omega$, and

$$\int_{\Omega \times \Omega} |F(z,w)|^2 e^{-2m_1\varphi(z) - 2m_2\varphi(w)} \, d\lambda(z) \, d\lambda(w) \leqslant C (= C_{\Omega \times \Omega}). \tag{4.5}$$

If $\{\sigma_j\}$ is an orthonormal basis in $H^2(\Omega, e^{-2m_1\varphi_{m_1}})$ and $\{\sigma'_k\}$ an orthonormal basis in $H^2(\Omega, e^{-2m_1\varphi_{m_2}})$ then one can easily check that $\{\sigma_j(z)\sigma'_k(w)\}$ is an orthonormal basis in $H^2(\Omega \times \Omega, e^{-2m_1\varphi_{m_1}(z) - 2m_2\varphi_{m_2}(w)})$. We may write

$$F(z,w) = \sum_{j,k} c_{jk} \sigma_j(z) \sigma'_k(w)$$

and by (4.5)

$$\sum_{j,k} |c_{jk}|^2 \leqslant C.$$

Therefore by the Schwarz inequality

$$|f(z)|^2 = |F(z,z)|^2 \leqslant C \sum_j |\sigma_j(z)|^2 \sum_k |\sigma'_k(z)|^2 = C\mathrm{e}^{2m_1\varphi_{m_1}(z)}\mathrm{e}^{2m_2\varphi_{m_2}(z)}$$

(using (4.4)). Since f was arbitrary, the theorem follows with $C_3 = (\log C)/2$. □

Corollary 4.4. *The sequence* $\varphi_{2^k} + C_3/2^{k+1}$ *is decreasing.*

It is an open problem if the whole sequence φ_m from Theorem 4.1 (perhaps modified by constants as in Corollary 4.4) is decreasing.

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