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AN ELEMENTARY PROOF OF THE MCCOY THEOREM

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Abstract. A simple proof of the McCoy theorem, which characterizes injective matrices over commutative rings is given. Only elementary facts like the Laplace formula and the Cramer theorem are used; no exterior algebra is needed. The McCoy theorem leads to the following property: in free modules over commutative rings linearly independent sets have no larger cardinality than sets of generators.

Here we give a simple proof of the McCoy theorem without using exterior algebra (compare [1], p.124, Theorem 3.5.1, [2], p.519, Proposition 12, p.524, Proposition 3). This theorem leads to interesting properties of free modules over commutative rings (Corollaries 1 and 2).

Let R be a commutative ring with identity. We will denote by $R^{n\times m}$ the set of all matrices over R of n rows and m columns. A matrix $A \in R^{n\times m}$ will be identified with the linear mapping of free R-modules:

$$R^m \ni b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \longmapsto Ab \in R^n.$$

The following two facts are well known:

1) (the Laplace formula) Let $A = [a_{ij}]_{i,j=1,\dots,n} \in R^{n\times n}$ be a square matrix. Then

$$\det A = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}M_{ij} \qquad i = 1,\dots,n;$$

where M_{ij} denotes the minor of rank $n-1$ obtained by suppressing in A the row of index i and the column of index j.

2) (the Cramer theorem) For every $A \in R^{n\times n}$ there exists $B \in R^{n\times n}$ such that $AB = BA = (\det A)I$ (I denotes the unit matrix).

THE McCOY THEOREM.

A matrix $A \in R^{n\times m}$ is a monomorphism if and only if $n \geq m$ and zero is the only element in R which annihilates all minors of maximal rank (i.e. of rank m) of the matrix A.

PROOF. First assume that the second condition is fulfilled and $Ab = 0$ for some $b \in R^m$. Let $M = \det \overline{A}$ be a minor of rank m of A. Since $\overline{A}b = 0$, from the Cramer theorem we have $Mb = 0$. Thus $b = 0$ and A is a monomorphism.

Conversely let $A = [a_{ij}]_{i=1,\ldots,n;\, j=1,\ldots,m}$ be a monomorphism. We may assume that $n \geq$m. Indeed: suppose that we have proved the theorem for square matrices. If $n < m$, then the square matrix

$$\begin{bmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nm} \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \in R^{m\times m}$$

is again a monomorphism and its determinant is equal to zero, which is contradictory to the above assumption.

Let $a \in R$ be such that $aM = 0$ for every minor M of rank m (i.e. of maximal rank). By inverse induction we will prove that for $k = 1, \ldots, m$ the following statement is true:

$(*)_k$ $\qquad aM = 0$ for all minors M of rank k.

It will complete the proof because from $(*)_1$ it follows that

$$aA = 0, \quad A\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = 0$$

and finally, from the fact that A is a monomorphism, $a = 0$.

We see that $(*)_m$ is true. Now assume that $(*)_{k+1}$ $(k = 1, \ldots, m-1)$ is and let M be a minor of rank k. We may assume that $M = \det((a_{ij})_{i,j=1,\ldots,k})$. Let M_j $(j = 1, \ldots, k+1)$ be a minor of rank k obtained by suppressing the column of index j in the matrix

$$\begin{bmatrix} a_{11} & \ldots & a_{1\,k+1} \\ \vdots & \ddots & \vdots \\ a_{k1} & \ldots & a_{k\,k+1} \end{bmatrix} \in R^{k\times(k+1)}$$

In particular $M_{k+1} = M$. Let

$$b: = a \begin{bmatrix} M_1 \\ -M_2 \\ \vdots \\ (-1)^{k+1} M_{k+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in R^m .$$

We will show that $Ab = 0$, i.e. that

$$\sum_{j=1}^{k+1} (-1)^j a_{ij} a M_j = 0 \qquad i = 1, \ldots, n.$$

Indeed, from the Laplace formula we have

$$(-1)^{k+1} \sum_{j=1}^{k+1} (-1)^j a_{ij} a M_j = a \det \begin{bmatrix} a_{11} & \ldots & a_{1\,k+1} \\ \vdots & \ddots & \vdots \\ a_{k1} & \ldots & a_{k\,k+1} \\ a_{i1} & \ldots & a_{k\,k+1} \end{bmatrix} .$$

If $i = 1, \ldots, k$ then the expression on the right is equal to zero as a determinant of a matrix with two identical rows. If $i = k+1, \ldots, n$, then it equals zero from the inductive assumption $(*)_{k+1}$. Thus $Ab = 0$ and, from the fact that A is a monomorphism, $b = 0$. In particular, $aM = aM_{k+1} = 0$. The theorem is proved.

COROLLARY 1.

Let M be a free R–module, L – a linearly independent set in M and G – a generating set. Then $\operatorname{card} L \leq \operatorname{card} G$.

PROOF.

Let B be a basis of M.

I) $\operatorname{card} B \leq \operatorname{card} G$.

Let J be a maximal ideal in R. Then R/J is a field and M/JM– a vector space over R/J. Let $\overline{B} := \{b + JM : b \in B\}$, $\overline{G} := \{g + JM : g \in G\}$. After checking that $\overline{B}$ is a basis of M/JM over R/J, $\operatorname{card} \overline{B} = \operatorname{card} B$ and $\operatorname{card} \overline{G} \leq \operatorname{card} G$ we will get $\operatorname{card} B = \operatorname{card} \overline{B} \leq \operatorname{card} \overline{G} \leq \operatorname{card} G$

II) $\operatorname{card} L \leq \operatorname{card} B$ if B is finite.

Direct consequence of the McCoy theorem .

III) $\operatorname{card} L \leq \operatorname{card} B$ if B is infinite.

Let $F(B)$ denote the set of all finite subsets of B. We have $\operatorname{card} F(B) = \operatorname{card} B$. For $C \in F(B)$ let $L_C := L \cap < C >$ ($< C >$ denotes the submodule of M generated by C). From II) we have $\operatorname{card} L_C \leq \operatorname{card} C < \aleph_0$. Hence

$$\operatorname{card} L = \operatorname{card} \bigcup_{C \in F(B)} L_C \leq \aleph_0 \operatorname{card} F(B) = \operatorname{card} B.$$

The proof is completed.

As a direct consequence of Corollary 1 we obtain

COROLLARY 2.
If N is a free submodule of a free R-module M, then $\dim N \leq \dim M$.

References

1. Balcerzyk S., Józefiak T., *Multiplicity and Homological Methods*, Horwood – PWN, Warszawa, 1989.
2. Bourbaki N., *Algebra I. Chapters 1–3*, Hermann – Addison – Wesley, London, 1974.

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