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Suita Conjecture from the One-dimensional Viewpoint

Zbigniew Błocki

Dedicated to the memory of Mikael Passare

Abstract. The Suita conjecture predicted the optimal lower bound for the Bergman kernel of a domain on the plane in terms of logarithmic capacity. It was recently proved as a special case of the optimal version of the Ohsawa–Takegoshi extension theorem. We present here a purely one-dimensional approach that should be suited to readers not interested in several complex variables.

Introduction

For a domain Ω in $\mathbb{C}$ by $A^2(\Omega)$ we denote the space of holomorphic square integrable functions in Ω. The Bergman kernel K_Ω is defined by the reproducing property

$$f(w) = \int_\Omega f\overline{K_\Omega(\cdot,w)}d\lambda, \quad f \in A^2(\Omega),\ w \in \Omega.$$

On the diagonal we have

$$K_\Omega(w,w) = ||K_\Omega(\cdot,w)||^2 = \sup\{|f(w)|^2 \colon f \in A^2(\Omega),\ ||f|| \le 1\}$$

where $||\cdot||$ denotes the L^2-norm. Suita [17] conjectured that

$$c_\Omega(w)^2 \le \pi K_\Omega(w,w) \tag{1}$$

where

$$c_\Omega(w) = \exp \lim_{z\to w}(G_\Omega(z,w) - \log|z-w|)$$

is the logarithmic capacity of the complement of Ω with respect to w. Here G_Ω is the Green function, it is the maximal negative function such that $G_\Omega(\cdot,w) - \log|\cdot -w|$ is harmonic in Ω (or $\equiv -\infty$).

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Ohsawa [15] was the first to notice that the right approach is to treat it an L^2-extension problem: one has to construct holomorphic f in Ω such that $f(w) = 1$ and $||f||^2 \leq \pi/c_\Omega(w)^2$. Using the methods of the original proof of the Ohsawa–Takegoshi extension theorem [16] he managed to show the estimate

$$c_\Omega(w)^2 \leq C\pi K_\Omega(w, w)$$

with $C = 750$. This was improved to $C = 2$ in [3] and to $C = 1.95388\ldots$ by Guan–Zhou–Zhu [11] who proved the extension theorem with this constant using an ODE with one unknown (see also [4]).

The estimate with $C = 1$ was established in [5] where also the optimal version of the Ohsawa–Takegoshi theorem was obtained. The main tool was the Hörmander L^2-estimate [12] for the $\bar{\partial}$-equation as well as some ideas of Chen [8] who was the first to show that the extension theorem (without an optimal constant) can be deduced directly from this estimate. One of the key steps was a solution of an ODE with two unknowns. Guan–Zhou [9, 10] later proved some generalizations of the extension theorem with optimal constant but similarly as in [5] the key was essentially the same ODE with two unknowns.

Two other proofs of the Suita conjecture were found afterwards. Both of them gave the estimate

$$K_\Omega(w, w) \geq \frac{1}{e^{-2t}\lambda(\{G_\Omega(\cdot, w) < t\})}, \tag{2}$$

where $t \leq 0$, from which (1) easily follows when $t \to -\infty$. The first from [6] used the tensor-power trick and thus effectively needed an arbitrarily high dimension in order to obtain this one-dimensional result. The second was due to Lempert [13] who noticed that (2) can be deduced from subharmonicity property of the Bergman kernel for sections of a pseudoconvex domain in $\mathbb{C}^2$, see [14] and [2]. This way one had to use two dimensions to get the Suita conjecture. One can add that it was shown in [7] using the isoperimetric inequality that the right-hand side of (2) is monotone in t.

Using some ideas of Berndtsson [1] and essentially following the approach of Guan–Zhou [9] we will give a self-contained one-dimensional proof of the Suita conjecture (1). We will obtain the same ODE as in [5]. It would be interesting to find such a one-dimensional proof of (2). As a by-product in Section 2 we present a new formula for the Bergman kernel on the diagonal as an extremal for a family of test functions.

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1. Proof of the Suita conjecture

It is well known that the Bergman kernel, Green function and logarithmic capacity converge locally uniformly as Ω_j is an increasing sequence of domains whose union is Ω. Without loss of generality we may therefore assume that Ω is bounded and has smooth boundary.

We will use the notation

$$\partial=\frac{\partial}{\partial z},\quad \bar\partial=\frac{\partial}{\partial\bar z}.$$

The following description of the Bergman kernel as a solution of the Dirichlet problem is well known, we present the proof for the sake of completeness:

Proposition 1. *For $w\in\Omega$ where Ω is a bounded domain in $\mathbb{C}$ with C^1 boundary, let v be the complex-valued harmonic function in Ω such that $v(z)=1/(\pi\overline{(z-w)})$ for $z\in\partial\Omega$. Then $K_\Omega(\cdot,w)=\partial v$.*

Proof. It is clear that ∂v is holomorphic and we have to show that the reproducing formula is satisfied. Take $f\in A^2(\Omega)$, by the approximating property we may assume that f is defined in a neighbourhood of $\bar\Omega$. By the Cauchy–Green formula

$$f(w)=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(z)}{z-w}dz=-\frac{i}{2}\int_{\partial\Omega}f\bar v dz=\int_\Omega f\overline{\partial v}d\lambda$$

and the result follows. □

For a real-valued $\varphi\in C^1(\Omega)$ we consider the weighted scalar product

$$\langle\alpha,\beta\rangle_\varphi=\int_\Omega\alpha\bar\beta e^{-\varphi}d\lambda$$

and the adjoint operator

$$\bar\partial^*_\varphi\alpha=-e^\varphi\partial(\alpha e^{-\varphi})=-\partial\alpha+\alpha\partial\varphi,$$

so that

$$\langle\bar\partial^*_\varphi\alpha,\beta\rangle_\varphi=\langle\alpha,\bar\partial\beta\rangle_\varphi,$$

provided that $\varphi,\alpha,\beta\in C^1(\bar\Omega)$ are such that on $\partial\Omega$ either $\alpha=0$ or $\beta=0$. We have

$$\bar\partial\bar\partial^*_\varphi\alpha=\bar\partial^*_\varphi\bar\partial\alpha+\alpha\partial\bar\partial\varphi.\tag{3}$$

To prove (1) assume for simplicity that $w=0$ and set

$$\alpha:=e^\varphi(1-\pi\bar z v),\tag{4}$$

where v is as in Proposition 1 and φ will be determined later. We have $\alpha=0$ on $\partial\Omega$, $\alpha(0)=e^{\varphi(0)}$ and

$$\bar\partial^*_\varphi\alpha=\pi\bar z K_\Omega(\cdot,0)e^\varphi.$$

Then

$$K_\Omega(0,0)=\frac{1}{\pi^2}\int_\Omega|\bar\partial^*_\varphi\alpha|^2\frac{e^{-2\varphi}}{|z|^2}d\lambda.\tag{5}$$

We will need the following:

Proposition 2. *Assume that $\mu,\varphi\in C^2(\bar\Omega)$ are real-valued and $\alpha\in C^1(\bar\Omega)$ is such that $\alpha=0$ on $\partial\Omega$. Then*

$$\int_\Omega\mu|\bar\partial^*_\varphi\alpha|^2e^{-\varphi}d\lambda=\int_\Omega\left[\mu|\bar\partial\alpha|^2+|\alpha|^2\left(\mu\partial\bar\partial\varphi-\partial\bar\partial\mu\right)+2\Re\left(\bar\alpha\bar\partial\mu\,\bar\partial^*_\varphi\alpha\right)\right]e^{-\varphi}d\lambda.$$

Proof. We have

$$\langle \mu\bar\partial^*_\varphi\alpha, \bar\partial^*_\varphi\alpha\rangle_\varphi = \langle \bar\partial\mu\, \bar\partial^*_\varphi\alpha, \alpha\rangle_\varphi + \langle \mu\bar\partial\bar\partial^*_\varphi\alpha, \alpha\rangle_\varphi$$

and by (3)

$$\langle \mu\bar\partial\bar\partial^*_\varphi\alpha, \alpha\rangle_\varphi = \langle \bar\partial^*_\varphi\bar\partial\alpha, \mu\alpha\rangle_\varphi + \langle \alpha\partial\bar\partial\varphi, \mu\alpha\rangle_\varphi.$$

Further,

$$\langle \bar\partial^*_\varphi\bar\partial\alpha, \mu\alpha\rangle_\varphi = \langle \bar\partial\alpha, \mu\bar\partial\alpha\rangle_\varphi + \langle \bar\partial\alpha, \alpha\bar\partial\mu\rangle_\varphi$$

and

$$\langle \bar\partial\alpha, \alpha\bar\partial\mu\rangle_\varphi = \langle \alpha, \bar\partial\mu\bar\partial^*_\varphi\alpha\rangle_\varphi - \langle \alpha, \alpha\partial\bar\partial\mu\rangle_\varphi. \qquad \square$$

We obtain the following version of the Nakano inequality:

Corollary 3. *Let α and φ be as in Proposition 2. Assume that both $\mu_1 \in C^2(\bar\Omega)$ and integrable μ_2 are positive. Then*

$$\int_\Omega (\mu_1+\mu_2)|\bar\partial^*_\varphi\alpha|^2 e^{-\varphi}d\lambda \ge \int_\Omega |\alpha|^2\left(\mu_1\partial\bar\partial\varphi - \partial\bar\partial\mu_1 - \frac{|\partial\mu_1|^2}{\mu_2}\right)e^{-\varphi}d\lambda.$$

Proof. By Proposition 2 and the Cauchy–Schwarz inequality

$$\int_\Omega \mu_1|\bar\partial^*_\varphi\alpha|^2e^{-\varphi}d\lambda \ge \int_\Omega \left[|\alpha|^2\left(\mu_1\partial\bar\partial\varphi - \partial\bar\partial\mu_1\right) - \frac{|\alpha\bar\partial\mu_1|^2}{\mu_2} - \mu_2|\bar\partial^*_\varphi\alpha|^2\right]e^{-\varphi}d\lambda. \ \square$$

By (5) we see that we should use Corollary 3 with $\mu_1+\mu_2 = e^{-\varphi}/|z|^2$. Denote $G = G_\Omega(\cdot,0)$ and set $\psi := 2G - \log|z|^2$, so that ψ is harmonic in Ω and $c_\Omega(0)^2 = e^{\psi(0)}$. We will look for

$$\varphi = \psi + \chi(-2G), \quad \mu_1 = e^{-\gamma(-2G)},$$

where $\chi(t)$ and $\gamma(t)$ defined for $t=-2G\ge 0$ will be determined later. Note that

$$\mu_2 = \frac{e^{-\varphi}}{|z|^2} - \mu_1 = e^{t-\chi} - e^{-\gamma}.$$

Using the fact that

$$\partial\bar\partial G = \frac{\pi}{2}\delta_0$$

we will obtain

$$\begin{aligned}
&\mu_1\partial\bar\partial\varphi - \partial\bar\partial\mu_1 - \frac{|\partial\mu_1|^2}{\mu_2}\\
&\quad = -\pi(\chi'+\gamma')e^{-\gamma}\delta_0 + 4\left(\chi''+\gamma'' - \frac{(\gamma')^2}{1-e^{\chi-\gamma-t}}\right)e^{-\gamma}|\partial G|^2\\
&\quad = -\pi\eta' e^{-\gamma}\delta_0 + 4\left(\eta'' - \frac{(\gamma')^2}{1-e^{\eta-2\gamma-t}}\right)e^{-\gamma}|\partial G|^2,
\end{aligned}$$

where $\eta = \chi+\gamma$. It is convenient to choose γ and η satisfying $-\eta' e^{-\gamma} = 1$ and

$$\eta'' - \frac{(\gamma')^2}{1-e^{\eta-2\gamma-t}} = 0$$

which is the same equation as in [5]. We can take the solutions obtained there:

$$\eta = -\log(t + e^{-t} - 1)$$
$$\gamma = -\log(t + e^{-t} - 1) + \log(1 - e^{-t}),$$

so that

$$\chi = -\log(1 - e^{-t})$$

and

$$\varphi = \psi - \log(1 - e^{2G}) = \psi - \log(1 - |z|^2 e^{\psi}).$$

By Corollary 3, since $\alpha(0) = e^{\varphi(0)} = c_\Omega(0)^2$,

$$\int_\Omega |\bar\partial^*_\varphi \alpha|^2 \frac{e^{-2\varphi}}{|z|^2} d\lambda \geq \pi c_\Omega(0)^2$$

and it is enough to use (5) to obtain (1). (Although we have used Corollary 3 for μ_1 which is not C^2 at the origin – in fact it is of the form $\mu_1 = -2G + \rho$ where ρ is smooth – by approximation it is clear that it holds also for such a function.)

2. A Formula for the Bergman kernel

Using similar methods as before we will prove the following result:

Theorem 4. *For a domain Ω in $\mathbb{C}$ and $w \in \Omega$ one has*

$$K_\Omega(w,w) = \frac{1}{\pi^2} \inf \left\{ \int_\Omega \frac{|\partial\alpha(z)|^2}{|z-w|^2} d\lambda(z) \colon \alpha \in C_0^\infty(\Omega),\ \alpha(w) = 1 \right\}. \qquad (6)$$

Proof. We may assume that $w = 0$. Take $\alpha \in C_0^\infty(\Omega)$ and $f \in A^2(\Omega)$ with $\alpha(0) = f(0) = 1$. Then $u := f/(\pi z)$ solves $\bar\partial u = \delta_0$ and

$$1 = |\alpha(0)|^2 = \left| \int_\Omega \bar\alpha\, \bar\partial u \right|^2 = \left| - \int_\Omega u\, \overline{\partial\alpha}\, d\lambda \right|^2 \leq \frac{1}{\pi^2} ||f||^2 \int_\Omega \frac{|\partial\alpha|^2}{|z|^2} d\lambda.$$

This gives $\leq$ in (6). To prove $\geq$ we first assume that Ω is bounded and has smooth boundary. Let v be harmonic in Ω and such that $v = 1/(\pi\bar z)$ on $\partial\Omega$. Then $\alpha := 1 - \pi\bar z v$ is smooth up to the boundary, vanishes there and $\alpha(0) = 1$. By Proposition 1 we have $\partial\alpha = -\pi\bar z K_\Omega(\cdot, 0)$ and it is enough to show that α can be well approximated by test forms. Let ρ be a defining function for Ω, so that $\Omega = \{\rho > 0\}$, and let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(t) = 0$ for $t \leq 1$ and $\chi(t) = 1$ for $t \geq 2$. One can easily show that for the test forms $\alpha_j := \chi(j\rho)\alpha$ one has

$$\int_\Omega \frac{|\partial\alpha_j|^2}{|z|^2} d\lambda \longrightarrow \int_\Omega \frac{|\partial\alpha|^2}{|z|^2} d\lambda$$

as $j \to \infty$. If Ω is arbitrary and $K_\Omega(0,0) < a$ then we can find $\Omega' \Subset \Omega$ with smooth boundary such that $K_{\Omega'}(0,0) < a$. By the previous part there exists $\alpha \in C_0^\infty(\Omega')$

such that $\alpha(0) = 1$ and

$$\frac{1}{\pi^2}\int_{\Omega'}\frac{|\partial\alpha|^2}{|z|^2}d\lambda < a.$$

This finishes the proof. □

Similarly, for any $\varphi \in C^1(\Omega)$ one can show

$$K_\Omega(0,0) = \frac{1}{\pi^2}\inf\left\{\int_\Omega |\bar\partial^*_\varphi\alpha|^2\frac{e^{-2\varphi}}{|z|^2}d\lambda\colon \alpha \in C_0^\infty(\Omega),\ \alpha(0) = e^{\varphi(0)}\right\}.$$

If Ω is bounded with smooth boundary and $\varphi \in C^1(\bar\Omega)$ then instead of test forms we can take $\alpha \in C^1(\bar\Omega)$ with $\alpha = 0$ on $\partial\Omega$ and α given by (4) realizes the infimum.

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Zbigniew Błocki
Uniwersytet Jagielloński
Instytut Matematyki
Łojasiewicza 6
PL-30-348 Kraków, Poland
e-mail: Zbigniew.Blocki@im.uj.edu.pl
umblocki@cyf-kr.edu.pl