Regularity of the Pluricomplex Green Function with Several Poles

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ABSTRACT. We show that if $\Omega$ is a $C^{2,1}$ smooth, strictly pseudoconvex domain in $\mathbb{C}^n$, then the pluricomplex Green function for $\Omega$ with several fixed poles and positive weights is $C^{1,1}$.

1. INTRODUCTION

If $\Omega$ is a bounded domain in $\mathbb{C}^n$, $p^1, \ldots, p^k \in \Omega$ are distinct, and $\mu_1, \ldots, \mu_k > 0$, then the corresponding pluricomplex Green function is given by

$$g = \sup B,$$

where

$$B = \{ v \in PSH(\Omega) \mid v < 0, \limsup_{z \to p^i} (u(z) - \mu_i \log |z - p^i|) < \infty, i = 1, \ldots, k \}.$$

One can show that $g \in B$, $g$ is a maximal plurisubharmonic (psh) function in $\Omega \setminus \{p^1, \ldots, p^k\}$, and

$$Mg = \frac{\pi^n}{n!2^n} \sum_i \mu_i \delta_{p^i}$$

(see [Le]), where $M$ is the complex Monge-Ampère operator. For smooth $u$

$$Mu = \det \left( \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right),$$

and by [De] $Mu$ can be well defined as a nonnegative Borel measure if $u \in PSH(\Omega)$ and $u$ is locally bounded near $\partial \Omega$.

In this paper we want to show the following regularity result.
**Theorem 1.1.** Assume that $\Omega$ is $C^{2,1}$ smooth and strictly pseudoconvex. Then $g \in C^{1,1}(\Omega \setminus \{p^1, \ldots, p^k\})$, and

$$|\nabla^2 g(z)| \leq \frac{C}{\min_i |z - p^i|^2}, \quad z \in \Omega \setminus \{p^1, \ldots, p^k\},$$

where $C$ is a constant depending only on $\Omega$, $p^1, \ldots, p^k$, $\mu_1, \ldots, \mu_k$.

One can treat it as a regularity result for the complex Monge-Ampère operator and indeed, this is the main tool in the proof. The obtained regularity is the best possible: as shown in [Co] and [EZ], the Green function for a ball with two poles and equal weights is not $C^2$ inside. In the case of one pole it is known from [BD] that the Green function need not be $C^2$ up to the boundary, but in this example it is not clear how regular the function is inside. Therefore, a full counterexample is still missing in this case.

The case $k = 1$ was treated in [Gu] and [Bł3]. In [Gu] the $C^{1,\alpha}$ regularity for $\alpha < 1$ was claimed. However, the proof contained an error (inequality (3.6) on p. 697 in [Gu] is false). Then in [Bł3], using some results from [Gu] and a method similar to the one used in [BT1] involving holomorphic automorphisms of a ball, the $C^{1,1}$ regularity was shown. Afterwards, in the correction to [Gu], a different method was used to show the $C^{1,\alpha}$ regularity.

Here we adapt the methods from [Gu] and [Bł3] for $k \geq 1$. This yields also a slightly different proof for $k = 1$, as instead of the lemma from [Bł3] we use a holomorphic mapping

$$z \mapsto z + \frac{(z_1 - p_1) \cdots (z_1 - p_k)}{(a_1 - p_1) \cdots (a_1 - p_k)}h$$

(in appropriate variables given by Lemma 3.2 below), which for $a \not\in \{p^1, \ldots, p^k\}$ and small $h \in \mathbb{C}^n$ fixes $p^i$ and maps $a$ to $a + h$.

To get an a priori estimate for the second derivative on the boundary, we follow the method from [CKNS] and prove Theorems 4.1 and 4.2 below. In the case of Theorem 4.2 we also use a modification of this method from [Gu]. We present the full proofs of Theorems 4.1 and 4.2 for two reasons: firstly, since given functions are constant on the boundary and their complex Monge-Ampère measure is also constant, the proofs are simpler than in the general setting, and secondly, we get a precise dependence of the a priori constants which was stated neither in [CKNS] nor in [Gu]. In fact, all quantitative estimates necessary to obtain the constant from Theorem 1.1 are included here. We only make use of the existence result – [Gu, Theorem 1.1] (it would even be enough to use [CKNS, Theorem 1] and Theorem 4.1 and 4.2 below instead).

By the way, we are also able to show the following regularity of $g$. 
Theorem 1.2. If $\Omega$ is hyperconvex, then $g$ is continuous as a function defined on the set

$$\{(z, p^1, \ldots, p^k, \mu_1, \ldots, \mu_k) \in \Omega \times \Omega^k \times (\mathbb{R}^+)^k \mid z \neq p^i + p^j \text{ if } i \neq j\},$$

where for $z \in \partial \Omega$ we set $g := 0$.

(Recall that $\Omega$ is called hyperconvex if there exists $\psi \in PSH(\Omega)$ with $\psi < 0$ and $\lim_{z \to \partial \Omega} \psi(z) = 0$.)

Theorem 1.3. Assume that

$$\limsup_{z \to \partial \Omega} \frac{|g(z)|}{\text{dist}(z, \partial \Omega)} < \infty.$$

Then

$$|\nabla g(z)| \leq \frac{C}{\min_i |z - p^i|}, \quad z \in \Omega \setminus \{p^1, \ldots, p^k\},$$

where $C$ is a constant depending only on $\Omega$, $p^1$, $\ldots$, $p^k$, $\mu_1$, $\ldots$, $\mu_k$.

Notation. If $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, then $x_i = \text{Re} z_i$, $y_i = \text{Im} z_i$. If $\zeta \in \mathbb{C}^n$, $|\zeta| = 1$, then by $\partial^m u(z)$ we will denote the $m$-th derivative of $u$ in direction $\zeta$ at $z$. For the partial derivatives we will use the notation

$$u_{x_i} = \frac{\partial u}{\partial x_i}, \quad u_{y_i} = \frac{\partial u}{\partial y_i}, \quad u_i = \frac{\partial u}{\partial z_i}, \quad u_{\bar{z}_i} = \frac{\partial u}{\partial \bar{z}_i}.$$

If we write

$$|\nabla u| \leq f \quad \text{in an open } D \subset \mathbb{C}^n,$$

where $f$ is locally bounded, nonnegative in $D$, then we mean that $u$ is locally Lipschitz and the inequality holds almost everywhere ($|\nabla u|$ makes then sense by the Rademacher theorem). If we write $dd^c u \geq dd^c |z|^2$, in fact it means exactly that $u - |z|^2$ is psh. When proving the existence of a constant depending only on given quantities, by $C_1, C_2, \ldots$ we will denote positive constants depending only on those quantities and call them under control.

2. Basic estimates

Given a bounded domain $\Omega$ in $\mathbb{C}^n$, distinct poles $p^1, \ldots, p^k \in \Omega$ and weights $\mu_1, \ldots, \mu_k > 0$ fix positive $R$, $r$, $m$, and $M$ so that for $i, j = 1, \ldots, k$

$$\Omega \subset B(p^i, R),$$

$$\tilde{B}(p^i, r) \subset \Omega \quad \text{and} \quad \tilde{B}(p^i, r) \cap \tilde{B}(p^j, r) = \emptyset,$$

$$m \leq \mu_i \leq M.$$
One can easily check the following estimates for $g$:
\[
\sum_i \mu_i \log \frac{|z - p^i|}{R} \leq g(z) < 0, \quad z \in \Omega,
\]
\[
\mu_i \log \frac{|z - p|}{R} - (k - 1)M \log \frac{R}{r} \leq g(z) \leq \mu_i \log \frac{|z - p|}{r}, \quad z \in B(p^i, r).
\]

For $\varepsilon$ with $0 < \varepsilon < r$, define
\[
\Omega^\varepsilon := \Omega \setminus \bigcup_i \bar{B}(p^i, \varepsilon),
\]
and
\[
g^\varepsilon := \sup \left\{ v \in PSH(\Omega) \mid v < 0, \quad v \mid_{\bar{B}(p^i, \varepsilon)} \leq \mu_i \log \frac{\varepsilon}{r}, \quad i = 1, \ldots, k \right\}.
\]

One can easily check that
\[
g^\varepsilon(z) \leq \mu_i \log \frac{\max \{|z - p^i|, \varepsilon\}}{r}, \quad z \in \bar{B}(p^i, r),
\]
\[
g^\varepsilon \in PSH(\Omega),
\]
\[
g \leq g^\varepsilon \leq \frac{\log(r/\varepsilon)}{\log(r/\varepsilon) + (k - 1)(M/m) \log(r/\varepsilon)} g \quad \text{in} \ \Omega^\varepsilon,
\]
$g^\varepsilon \downarrow g^0 := g$ as $\varepsilon \downarrow 0$, and the convergence is locally uniform in $\Omega \setminus \{p^1, \ldots, p^k\}$.

**Proposition 2.1.** Assume that $\Omega$ is $C^\infty$ smooth and strictly pseudoconvex. Then there exists $r_0$ depending only on $k$, $r$, $R$, $m$, and $M$, $0 < r_0 \leq r$, such that for $\varepsilon$ with $0 < \varepsilon < r_0$ we can find $v \in PSH(\Omega) \cap C^\infty(\bar{\Omega})$ with $dd^c v \geq dd^c |z|^2$ in $\Omega$, $v = 0$ on $\partial \Omega$, and for $i = 1, \ldots, k$
\[
\mu_i \log \frac{\varepsilon}{r} \leq v(z) \leq \mu_i \log \frac{|z - p^i|}{r} \quad \text{if} \ \varepsilon \leq |z - p^i| \leq r.
\]

**Proof.** Set
\[
w(z) := \sum_i \mu_i \log \frac{|z - p^i|}{R} + |z - p^i|^2 - R^2,
\]
so that $w < 0$ on $\bar{\Omega}$, $dd^c v \geq dd^c |z|^2$, and $w < \mu_i \log(\varepsilon/r)$ on $\partial B(p^i, \varepsilon)$. On the other hand, for $z \in \partial B(p^i, r)$ we have
\[
w(z) \geq kM \log \frac{r}{R} + r^2 - R^2 > \mu_i \log \frac{\varepsilon}{r} + |z - p^i|^2 - \varepsilon^2,
\]
provided that $\varepsilon$ is such that
\[
m \log \frac{\varepsilon}{r} - \varepsilon^2 < kM \log \frac{r}{R} - R^2.
\]

Similarly as in [Bł2], let $\chi : \mathbb{R} \to \mathbb{R}$ be $C^\infty$ smooth and such that
\[
\begin{align*}
\chi(t) &= 0, \quad t \leq -1, \\
\chi(t) &= t, \quad t \geq 1, \\
0 &\leq \chi'(t) \leq 1, \quad t \in \mathbb{R}, \\
\chi''(t) &\geq 0, \quad t \in \mathbb{R}.
\end{align*}
\]

For $x, y \in \mathbb{R}$ set
\[
f_j(x, y) := x + \frac{1}{j} \chi(j(y - x)),
\]
so that
\[
f_j(x, y) = \max\{x, y\} \quad \text{if} \quad |x - y| \geq \frac{1}{j}.
\]

If $u$, $v$ are psh functions with $ddc u$, $ddc v \geq ddc |z|^2$, then
\[
ddc f_j(u, v) \geq (1 - \chi'(j(v - u)))ddc u + \chi'(j(v - u))ddc v \geq ddc |z|^2.
\]

Let $\psi$ be a defining function for $\Omega$. If we choose $j$, $A$ sufficiently big, then the function
\[
v(z) = \begin{cases} 
  f_j \left( w(z), \mu_i \log \frac{\varepsilon}{r} + |z - p_i|^2 - \varepsilon^2 \right), & z \in \bigcup_i \bar{B}(p_i, r), \\
  f_j(w(z), A \psi(z)), & z \in \bar{\Omega} \setminus \bigcup_i \bar{B}(p_i, r)
\end{cases}
\]
has all the required properties. $\square$

Note that if $k = 1$, then we may choose $r_0 = r$ in Proposition 2.1.

**Proof of Theorem 1.2.** By (2.2) $g^\varepsilon \to g$ locally uniformly on the set (1.1) as $\varepsilon \to 0$. It is thus enough to show that for a fixed small $\varepsilon$, $g^\varepsilon$ is continuous as a function defined on
\[
\bar{\Omega} \times \{(p^1, \ldots, p^k) \in \Omega^k \mid \text{dist}(p^i, \partial \Omega) > \varepsilon, \quad |p^i - p^j| > 2\varepsilon \text{ if } i \neq j\} \times (\mathbb{R}_+)^k.
\]

Let $p^{i,j} \to p^i$, $\mu_{i,j} \to \mu_i$ as $j \to \infty$, $i = 1, \ldots, k$, and
\[
g^\varepsilon_j := \sup \left\{ v \in PSH(\Omega) \mid v < 0, \quad v|_{\bar{B}(p^{i,j}, \varepsilon)} \leq \mu_{i,j} \log \frac{\varepsilon}{r} \right\}.
\]
Note that if \( 0 < \varepsilon < r_0 \) and \( j \) is big enough, then by Proposition 2.1 applied to a ball containing \( \Omega \) we have \( \lim_{z \to \partial B(p^i, \varepsilon)} g^f_j(z) = \mu_i \log(\varepsilon/r) \). Moreover, \( \lim_{z \to \partial \Omega} g^f_j(z) = 0 \), since \( \Omega \) is hyperconvex. Therefore, by a result from [Wa] (see also [Bł1, Theorem 1.5]), \( g^f_j \) is continuous on \( \hat{\Omega} \).

To finish the proof it is enough to show that \( g^f_j \to g^f \) uniformly as \( j \to \infty \) in \( \hat{\Omega} \). Fix \( c > 0 \). For \( z \in \hat{B}(p^i, \varepsilon) \) and \( j \) big enough, by (2.1) we have

\[
g^f_j(z) \leq \mu_{i,j} \log \frac{\max\{|z - p^{i,j}|, \varepsilon\}}{r} \leq \mu_{i,j} \log \frac{\varepsilon + |p^i - p^{i,j}|}{r} \leq \mu_i \log \frac{\varepsilon}{r} + c,
\]

whereas for \( z \in \hat{B}(p^{i,j}, \varepsilon) \)

\[
g^f(z) \leq \mu_i \log \frac{\max\{|z - p^{i}|, \varepsilon\}}{r} \leq \mu_i \log \frac{\varepsilon + |p^i - p^{i,j}|}{r} \leq \mu_{i,j} \log \frac{\varepsilon}{r} + c.
\]

Thus for those \( j \)

\[
g^f - c \leq g^f_j \leq g^f + c \quad \text{on } \hat{\Omega},
\]

and the theorem follows.

In the proof of Theorem 1.1 we will also need to approximate \( g^f \). If \( 0 \leq \varepsilon < r \) and \( 0 \leq \delta \leq 1 \), define

\[
g^{\varepsilon,\delta} := \sup \{ v \in \text{PSH} \cap L^\infty(\Omega) \mid v \leq g^f, Mv \geq \delta \text{ in } \Omega^\varepsilon \}.
\]

Note that \( g^{\varepsilon,\delta} \) is increasing in \( \varepsilon \) and decreasing in \( \delta \). We also have

(2.3) \( g^f + \delta(|z - p^i|^2 - R^2) \leq g^{\varepsilon,\delta} \leq g^f \).

**Proposition 2.2.** \( g^{\varepsilon,\delta} \in \text{PSH}(\Omega), Mg^{\varepsilon,\delta} = \delta \text{ in } \Omega^\varepsilon \). If \( \Omega \) is hyperconvex and \( 0 < \varepsilon < r_0 \), then \( g^{\varepsilon,\delta} \) is continuous on \( \hat{\Omega} \). If \( \Omega \) is \( C^\infty \) smooth and strictly pseudoconvex, \( 0 < \varepsilon < r_0 \) and \( 0 < \delta \leq 1 \), then \( g^{\varepsilon,\delta} \in C^\infty(\Omega^\varepsilon) \).

**Proof.** We use standard procedures. Let

\[
\mathcal{B} = \{ v \in \text{PSH}(\Omega) \mid v \leq g^f, Mv \geq \delta \text{ in } \Omega^\varepsilon \}.
\]

By the Choquet lemma there exists a sequence \( v_j \in \mathcal{B} \) such that \( (g^{\varepsilon,\delta})^* = (\sup_j v_j)^* \). (\( u^* \) denotes the upper semicontinuous regularization of \( u \).) If \( w_j = \max\{v_1, \ldots, v_j\} \), then \( Mw_j \geq \delta \) in \( \Omega^\varepsilon \) (see e.g. [Bł2]) and thus \( w_j \in \mathcal{B} \). Therefore \( w_j \uparrow (g^{\varepsilon,\delta})^* \) almost everywhere, and by the approximation theorem from [BT2] \( M(g^{\varepsilon,\delta})^* \geq \delta \) in \( \Omega^\varepsilon \). We conclude that \( g^{\varepsilon,\delta} \in \text{PSH}(\Omega) \) and \( Mg^{\varepsilon,\delta} \geq \delta \) in \( \Omega^\varepsilon \). The balayage procedure gives \( Mg^{\varepsilon,\delta} = \delta \) in \( \Omega^\varepsilon \).
Now assume that $\Omega$ is hyperconvex and $0 < \varepsilon < r_0$. By [BH1] there exists $\psi \in PSH(\Omega) \cap C(\Omega)$ with $\psi = 0$ on $\partial \Omega$ and $M \psi \geq 1$ in $\Omega$. For $A$ big enough

$$A\psi \leq g^{\varepsilon,\delta} \leq 0 \quad \text{in } \Omega.$$  

Let $v$ be given by Proposition 2.1 applied to a ball containing $\Omega$. Then

$$v(z) \leq g^{\varepsilon,\delta}(z) \leq \mu \log \frac{|z - p|}{r} \quad \text{if } \varepsilon \leq |z - p| \leq r.  \quad (2.4)$$

For small $h \in \mathbb{C}^n$ and $z \in \Omega$ with $|h| < \text{dist}(z, \partial \Omega) < 2|h|$ we have

$$|g^{\varepsilon,\delta}(z + h) - g^{\varepsilon,\delta}(z)| \leq C(|h|).$$

By the comparison principle (see [BT2]) applied to $g^{\varepsilon,\delta}$ and $g^{\varepsilon,\delta}(\cdot + h)$, the above inequality holds for all $z$ with $\text{dist}(z, \partial \Omega) > |h|$. By (2.4) and (2.5)

$$\lim_{h \to 0} C(|h|) = 0,$$

which means that $g^{\varepsilon,\delta}$ is continuous.

The last part of the proposition follows from Proposition 2.1 and [Gu, Theorem 1.1].

3. GRADIENT ESTIMATES

Theorem 1.3 will follow immediately from the next result applied to $\delta = 0$.

**Theorem 3.1.** Fix $0 \leq \delta \leq 1$. Assume that

$$\limsup_{z \to \partial \Omega} \frac{|g^{0,\delta}(z)|}{\text{dist}(z, \partial \Omega)} \leq B < \infty.$$

Then for $\varepsilon$ satisfying Proposition 2.1 we have

$$|\nabla g^{\varepsilon,\delta}(z)| \leq \frac{C}{\min|z - p|}, \quad z \in \Omega,$$

where $C$ is a constant depending only on $n, k, R, r, m, M,$ and $B$.

The assumption of Theorem 3.1 is satisfied uniformly for $\delta \leq 1$ for example, if $\Omega$ is smooth and strictly pseudoconvex.

**Proof of Theorem 3.1.** Let $\rho > 0$ be such that

$$-g^{\varepsilon,\delta}(z) \leq -g^{0,\delta}(z) \leq 2B \text{dist}(z, \partial \Omega) \quad \text{if } \text{dist}(z, \partial \Omega) \leq \rho.$$
For \( h \) sufficiently small
\[
\varphi^{\varepsilon,\delta}(z + h) - \varphi^{\varepsilon,\delta}(z) \leq 2B|h| \quad \text{if } \text{dist}(z, \partial\Omega) = |h|,
\]
and, since by Proposition 2.1
\[
\mu_1 \log \frac{\varepsilon}{r} \leq \varphi^{\varepsilon,\delta}(z) \leq \mu_1 \log \frac{|z - p^i|}{r} \quad \text{if } \varepsilon \leq |z - p^i| \leq r,
\]
we have
\[
\varphi^{\varepsilon,\delta}(z + h) - \varphi^{\varepsilon,\delta}(z) \leq \mu_1 \log \frac{|z - p^i + h|}{\varepsilon} \leq 2\frac{H_1}{\varepsilon}|h|,
\]
if \( z \in \partial B(p^i, \varepsilon + |h|), \ i = 1, \ldots, k. \)

From the comparison principle we get
\[
\varphi^{\varepsilon,\delta}(z + h) - \varphi^{\varepsilon,\delta}(z) \leq 2\max \left\{ B, \frac{M}{\varepsilon} \right\} |h| \quad \text{if } |h| \leq \min\{\rho, \text{dist}(z, \partial\Omega^\varepsilon)\},
\]
and thus
\[
(3.1) \quad |\nabla \varphi^{\varepsilon,\delta}| \leq \frac{C_1}{\varepsilon} \quad \text{in } \Omega^\varepsilon.
\]

We will need a lemma.

**Lemma 3.2.** There exists a constant \( \tilde{C} = \tilde{C}(k, n) \) such that for given \( p^1, \ldots, p^k \in \mathbb{C}^n, a \in \mathbb{C}^n \setminus \{p^1, \ldots, p^k\} \) we can orthonormally change variables in \( \mathbb{C}^n \) so that
\[
|a - p^i| \leq \tilde{C}|a_1 - p^i_1|, \quad i = 1, \ldots, k.
\]

**Proof.** By \( S \) denote the unit sphere in \( \mathbb{C}^n \). We have to show that there exists \( b \in S \) such that
\[
|a - p^i| \leq \tilde{C}|(a - p^i, b)|, \quad i = 1, \ldots, k,
\]
that is,
\[
\left| \frac{(a - p^i)}{|a - p^i|}, b \right| \geq \frac{1}{\tilde{C}}.
\]
Define
\[
\tilde{C} := \frac{1}{\min_{S^k} f},
\]
where

\[ f(\zeta^1, \ldots, \zeta^k) := \max_{b \in S} \min_{i} |\langle \zeta^i, b \rangle| \]

is a continuous function on \( S^k \). It remains to show that \( f > 0 \) on \( S^k \). Fix \( \zeta^1, \ldots, \zeta^k \in S \) and define \( K_i := \{ b \in S \mid \langle b, \zeta^i \rangle = 0 \}, \ i = 1, \ldots, k \). Then \( \bigcup_i K_i \neq S \), and thus for \( b \in S \setminus \bigcup_i K_i \) we have

\[ f(\zeta^1, \ldots, \zeta^k) \geq \min_{i} |\langle \zeta^i, b \rangle| > 0. \]

End of proof of Theorem 3.1. Fix \( a \in \Omega^e \) and choose variables as in Lemma 3.2. Set

\[ P(\lambda) := (\lambda - p_{11}) \cdots (\lambda - p_{1k}), \]

so that

\[ \frac{|P(z_1)|}{|P(a_1)|} \leq C_2 \frac{\max_i |z - p_i|}{\min_i |a - p_i|} \leq \frac{C_3}{\min_i |a - p_i|}, \ z \in \Omega. \]

For \( h \) sufficiently small let

\[ \Omega'' := \left\{ z \in \Omega \mid z + \frac{P(z_1)}{P(a_1)} h \in \Omega \right\} \]

and

\[ \Omega' := \Omega'' \setminus \bigcup_i B(p^i, \epsilon + \epsilon'), \]

where

\[ \epsilon' = \min\{\epsilon, r - \epsilon, \text{dist}(a, \partial \Omega^e), \rho\}. \]

Set

\[ u(z) := g^{\epsilon, \delta} \left( z + \frac{P(z_1)}{P(a_1)} h \right) + \frac{C_4}{\min_i |a - p_i|} (|z - p_i|^2 - R^2) |h|, \]

so that if \( C_4 \) is big enough, then

\[ M v \geq \left| 1 + \frac{P'(z_1)}{P(a_1)} h_1 \right|^2 \delta + \frac{C_4}{\min_i |a - p_i|} |h| \geq \delta. \]
For \( z \in \partial \Omega^{''} \) we have
\[
v(z) - g^{\varepsilon, \delta}(z) \leq 2B \text{dist}(z, \partial \Omega) \leq 2B \frac{C_3}{\min \{ |a - p|^i \} |h|},
\]
whereas for \( z \in \partial B(p^i, \varepsilon + \varepsilon') \)
\[
v(z) - g^{\varepsilon}(z) \leq \frac{C_1 |P(z)|}{\varepsilon |P(a_1)|} |h| \leq C_1 C_2 \frac{\varepsilon + \varepsilon'}{\varepsilon \min \{ |a - p|^i \} |a - p|^i} \leq \frac{C_5}{\min \{ |a - p|^i \} |h|}.
\]
Therefore, the comparison principle gives
\[
g^{\varepsilon}(a + h) - g^{\varepsilon}(a) \leq \frac{C_6}{\min \{ |a - p|^i \} |h|} \quad \text{if} \quad |h| \leq \varepsilon' \frac{\min \{ |a - p|^i \} |a - p|^i}{C_3},
\]
and the theorem follows. \( \Box \)

4. ESTIMATES OF THE SECOND DERIVATIVE

Our goal will be to estimate \( |\nabla^2 g^{\varepsilon, \delta}| \) for small \( \varepsilon, \delta \). First, we need such an estimate on \( \partial \Omega^{''} \). We will follow the method from [CKNS] (see also [Gu]). We shall prove two theorems.

**Theorem 4.1.** Let \( \Omega \) be a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) and \( \psi \) a \( C^\infty \) psh defining function for \( \Omega \). Assume that \( \ddbar^\varepsilon \psi \geq \ddbar \varepsilon \varepsilon |z|^2 \) and that there are positive constants \( A, a \) such that
\[
|\psi|, |\nabla \psi|, |\nabla^2 \psi|, |\nabla^3 \psi| \leq A \quad \text{on} \quad \Omega,
|\nabla \psi| \geq a \quad \text{on} \quad \partial \Omega.
\]
For \( \rho > 0 \) denote \( U = \{ z \in \mathbb{C}^n \mid \text{dist}(z, \partial \Omega) < \rho \} \). Let \( u \in \text{PSH}(\Omega \cap U) \cap C^\infty(\bar{\Omega} \cap U) \) be such that \( u = 0 \) on \( \partial \Omega \) and \( u < 0 \), \( Mu = \delta \) in \( \Omega \cap U \), where \( 0 < \delta \leq \delta_0 \). Assume also that there are positive constants \( b, B \) such that
\[
|\nabla u| \geq b \quad \text{on} \quad \partial \Omega,
|\nabla u| \leq B \quad \text{on} \quad \bar{\Omega} \cap U.
\]
Then there is a constant \( C = C(n, \rho, a, A, b, B, \delta_0) \) such that
\[
|\nabla^2 u| \leq C \quad \text{on} \quad \partial \Omega.
\]

**Theorem 4.2.** Fix \( \alpha > 1 \) and let \( \Omega = \{ z \in \mathbb{C}^n \mid 1 < |z| < \alpha \} \). Assume that \( u \in \text{PSH}(\Omega) \cap C^\infty(\bar{\Omega}) \) is such that \( u = 0 \) on \( \partial B_1 \) (\( B_\alpha = B(0, \alpha) \)), \( u > 0 \),
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\[ M \mathbf{u} = \delta > 0 \text{ in } \Omega. \] Suppose, moreover, that there are positive constants \( \beta, b, B \) such that

\[
\begin{align*}
  u &\geq \beta \quad \text{on } \partial B_{\alpha}, \\
  |\nabla u| &\geq b \quad \text{on } \partial B_{1}, \\
  |\nabla u| &\leq B \quad \text{on } \bar{\Omega}.
\end{align*}
\]

Then there exist positive constants \( \delta_0 = \delta_0(n, \alpha, \beta) \) and \( C = C(n, \alpha, \beta, b, B) \) such that if \( 0 < \delta \leq \delta_0 \), we have

\[ |\nabla^2 u| \leq C \quad \text{on } \partial B_{1}. \]

**Proof of Theorem 4.1.** Fix \( z_0 \in \partial \Omega \). We may assume that \( N_{z_0} = (0, \ldots, 0, 1) \), so that \( \partial N_{z_0} = \partial / \partial x_n \). Since both \( \psi \) and \( u \) are \( C^\infty \) defining functions for \( \Omega \), there exists a \( C^\infty \) function \( v \), defined in a neighborhood of \( \partial \Omega \), such that \( u = v \psi \) and \( v > 0 \) on \( \bar{\Omega} \cap U \). Therefore, if \( t, s \in \{ x_1, y_{\bar{1}}, \ldots, x_{n-1}, y_{n-1}, y_n \} \), then

\[
(4.1) \quad u_{ts}(z_0) = \frac{u_{x_n}(z_0)\psi_{ts}(z_0)}{\psi_{x_n}(z_0)}
\]

and thus

\[
(4.2) \quad |u_{ts}(z_0)| \leq C_1.
\]

Suppose now that we know that

\[
(4.3) \quad |u_{tx_n}(z_0)| \leq C_2,
\]

and we want to estimate \( |u_{x_nx_n}(z_0)| \). We have

\[ u_{x_nx_n} = 4u_{n\bar{n}} - u_{y_ny_n}, \]

and by (4.1), (4.2), (4.3), and since \( dd^c \psi \geq dd^c |z|^2 \),

\[
\delta_0 \geq \delta = \det(u_{ij}(z_0)) = u_{n\bar{n}}(z_0) \left( \frac{a}{A} \right)^{n-1} - C_3.
\]

It thus remains to show (4.3). For \( z \in \bar{\Omega} \) we have

\[
\psi_{x_n}(z) = \text{Re} \left( \nabla \psi(z), \frac{\nabla \psi(z_0)}{|\nabla \psi(z_0)|} \right) \geq |\nabla \psi(z_0)| - A|z - z_0| \geq a - A|z - z_0|.
\]
On $\tilde{\Omega} \cap \tilde{B}(z_0, \tilde{\rho})$ define

$$ T := u_t - \frac{\psi_t}{\psi_{x_n}} u_{x_n}, $$

so that

$$ T = 0 \quad \text{on } \partial \Omega \cap \tilde{B}(z_0, \tilde{\rho}). $$

We have

$$ T_{x_n}(z_0) = u_{tx_n}(z_0) - \frac{\psi_{tx_n}(z_0)}{\psi_{x_n}(z_0)} u_{x_n}(z_0), $$

and thus it is enough to prove that

$$ |T_{x_n}(z_0)| \leq C_4. $$

Set $f := \psi_t / \psi_{x_n}$; then

$$ |\nabla f|, \quad |\nabla^2 f| \leq C_5 \quad \text{in } \hat{\Omega} \cap \tilde{B}(z_0, \tilde{\rho}). $$

Since det$(u_{ij})$ is constant, one can show that

$$ u^{ij} u_{ij} = u^{ij} u_{ij x_n} = 0. $$

(Here $(u^{ij})$ denotes the inverse transposed matrix of $(u_{ij})$.) Hence, we can compute

$$ u^{ij} T_{ij} = -u_{x_n} u^{ij} f_{ij} - 2 \Re u^{ij} u_{ix_n} f_j = -u_{x_n} u^{ij} f_{ij} - 2 f_{x_n} - 2 \Im u^{ij} u_{iy_n} f_j. $$

Since

$$ u^{ij} (u^{2}_{y_n})_{ij} = 2 u^{ij} u_{iy_n} u_{jy_n}, $$

the Schwarz inequality and (4.5) give

$$ u^{ij} \left( \pm T + \frac{1}{2} u^{2}_{y_n} \right)_{ij} \geq -u_{x_n} u^{ij} f_{ij} + 2 f_{x_n} - u^{ij} f_i f_j \geq -C_6 \left( \sum_i u^{ii} + 1 \right). $$

On $\partial \Omega$ we have $u_{y_n} = u_{x_n} \psi_{y_n} / \psi_{x_n}$, and thus by (4.4)

$$ \left| \pm T + \frac{1}{2} u^{2}_{y_n} \right| \leq C_7 |z - z_0|^2, \quad z \in \partial \Omega \cap \tilde{B}(z_0, \tilde{\rho}). $$
Moreover,
\[ \left| \pm T + \frac{1}{2} u_{\gamma n}^2 \right| \leq C_8 \quad \text{in } \Omega \cap \bar{B}(z_0, \tilde{\rho}), \]
and we obtain that if \( w = \pm T + \frac{1}{2} u_{\gamma n}^2 - C_9 |z - z_0|^2 \), where \( C_9 \) is big enough, then \( w \leq 0 \) on \( \partial (\Omega \cap B(z_0, \tilde{\rho})) \), and
\[ u^{ij} w_{ij} \geq -C_{10} \left( \sum_i u^{ii} + 1 \right). \]

Therefore, if \( C_{11} \) and \( C_{12} \) are big enough, then \( w + C_{11} \psi + C_{12} u \leq 0 \) on \( \partial (\Omega \cap B(z_0, \tilde{\rho})) \) and \( u^{ij} (w + C_{11} \psi + C_{12} u)_{ij} \geq 0 \) in \( \Omega \cap B(z_0, \tilde{\rho}) \). By the maximum principle
\[ w + C_{11} \psi + C_{12} u \leq 0 \quad \text{in } \Omega \cap B(z_0, \tilde{\rho}), \]
and thus
\[ |T_{x_n}(z_0)| \leq C_{11} A + C_{12} B. \]

**Proof of Theorem 4.2.** Set
\[ \psi(z) = \lambda (|z|^2 - 1), \]
where \( \lambda = \beta / (\alpha^2 - 1) \), so that \( \psi \leq u \) in \( \Omega \) for \( \delta \) sufficiently small. We now follow the proof of Theorem 4.1. Fix \( z_0 \in \partial B_1 \), we may assume that \( z_0 = (0, \ldots, 0, 1) \). We may reduce the problem to the estimate
\[ |u_{tx_n}(z_0)| \leq C_1. \]
Similarly as before we get that if \( w = \pm T + \frac{1}{2} u_{\gamma n}^2 - C_2 |z - z_0|^2 \), where \( C_2 \) is big enough, then
\[ u^{ij} w_{ij} \geq -C_3 \left( \sum_i u^{ii} + 1 \right) \quad \text{in } \Omega \cap B(z_0, 1), \]
and \( w \leq 0 \) on \( \partial (\Omega \cap B(z_0, 1)) \).

Now by the inequality between arithmetic and geometric means we have
\[ u^{ij} (\psi - u)_{ij} \geq \lambda \sum_i u^{ii} - n \geq \frac{\lambda}{2} \sum_i u^{ii} + n \left( \frac{\lambda}{2 \delta^2 n} - 1 \right) \geq \frac{\lambda}{2} \left( \sum_i u^{ii} + 1 \right), \]
for \( \delta \) small enough. Thus
\[ u^{ij} (w + C_4 (\psi - u))_{ij} \geq 0 \quad \text{in } \Omega \cap B(z_0, 1) \]
if $C_4$ is sufficiently big, and by the maximum principle we conclude that

$$|T_{x_n}(z_0)| \leq C_4 B.$$ 

\[\square\]

Proof of Theorem 1.1. Let $\psi$ be a $C^{2,1}$ defining function for $\Omega$ with $dd^c \psi \geq dd^c |z|^2$ in $\Omega$ and

$$|\psi|, |\nabla \psi|, |\nabla^2 \psi|, |\nabla^3 \psi| \leq A \quad \text{on } \bar{\Omega},$$

$$|\nabla \psi| > a \quad \text{on } \partial \Omega,$$

for some positive $a$ and $A$. We can find $\tilde{r} > 0$ such that for every $z_0 \in \partial \Omega$ there exists a ball $B(z_1, 2\tilde{r})$, contained in $\Omega$ and tangent to $\partial \Omega$ at $z_0$. Then

$$g(z) \leq -\frac{\gamma}{\log 2} \log \frac{|z - z_1|}{2\tilde{r}} \quad \text{if } \tilde{r} \leq |z - z_1| \leq 2\tilde{r},$$

where

$$\gamma = \max_{\text{dist}(z, \partial \Omega) \geq \tilde{r}} g(z).$$

Therefore we can find $b$ with

$$\liminf_{z \to \partial \Omega} \frac{|g(z)|}{\text{dist}(z, \partial \Omega)} > b > 0.$$

Let $\psi_j = \psi \ast \rho_{1/j}$ be the standard regularization of $\psi$ and let $\Omega_j = \{ \psi_j < 0 \}$. If $j$ is big enough, then the constants $A$, $a$, and $b$ are good also for $\psi_j$ and $\Omega_j$. Thus, we may assume that $\psi$ (and thus $\Omega$) is $C^\infty$, provided that we prove that the constant in Theorem 1.1 depends only on $n$, $k$, $r$, $R$, $m$, $M$, $A$, $a$, and $b$.

By Proposition 2.2, $g^{\epsilon, \delta} \in C^\infty(\bar{\Omega})$ if $0 < \epsilon < r_0$, $0 < \delta \leq 1$. It is enough to show that for small positive $\epsilon$ and $\delta$ we have

$$|\nabla^2 g^{\epsilon, \delta}(z)| \leq \frac{C_1}{\min_i |z - p_i|^2}, \quad z \in \bar{\Omega}.$$

Since $|\nabla g^{\epsilon, \delta}| \geq b$ on $\partial \Omega$, by Theorems 3.1 and 4.1 we have

(4.6) $$|\nabla^2 g^{\epsilon, \delta}| \leq C_2 \quad \text{on } \partial \Omega.$$ 

For $|w| \geq 1$ and fixed $i = 1, \ldots, k$ set

$$u(w) := g^{\epsilon, \delta}(p_i + \epsilon w) - \mu_i \log \frac{\epsilon}{r}.$$
By (2.2) and (2.3)
\[ u(w) \geq \mu_1 \log |w| - C_3. \]
Thus, if \( \alpha \) is so big that \( \beta := m \log \alpha - C_3 > 0 \), then for sufficiently small \( \varepsilon \), \( u \geq \beta \) on \( \partial B_\alpha \). Moreover, \( g^{\varepsilon,\delta} \geq -C_4 \) on \( \partial B(p^i, r) \). Thus by the comparison principle, for sufficiently small \( \varepsilon \) we have
\[ \frac{\mu_i}{2} \log \frac{|z - p^i|}{r} + \frac{\mu_i}{2} \log \frac{\varepsilon}{r} + |z - p^i|^2 - \varepsilon^2 \leq g^{\varepsilon,\delta}(z) \text{ if } \varepsilon \leq |z - p^i| \leq r. \]
Therefore
\[ |\nabla g^{\varepsilon,\delta}| \geq \frac{\mu_i}{2\varepsilon} \text{ on } \partial B(p^i, \varepsilon), \]
and \( |\nabla u| \geq \mu_i/2 \) on \( \partial B_1 \). From Theorem 4.2 it follows that for \( \delta \) small enough
\[ |\nabla^2 u| \leq C_5 \text{ on } \partial B_1, \]
which means that
\[ (4.7) \quad |\nabla^2 g^{\varepsilon,\delta}| \leq \frac{C_5}{\varepsilon^2} \text{ on } \partial B(p^i, \varepsilon). \]

The rest of the proof will be a compilation of the methods from [Bł3] and from the proof of Theorem 3.1. Fix \( a \in \Omega \setminus \{p^1, \ldots, p^k\} \). From the fact that \( g^{\varepsilon,\delta} \) is psh it follows that
\[ (4.8) \quad |\nabla^2 g^{\varepsilon,\delta}(a)| = \limsup_{h \to 0} \frac{g^{\varepsilon,\delta}(a + h) + g^{\varepsilon,\delta}(a - h) - 2g^{\varepsilon,\delta}(a)}{|h|^2}. \]
Let \( P \) be as in the proof of Theorem 3.1 and let \( \Omega'' \subset \Omega' \subset \Omega, \varepsilon' > 0 \). For \( z \in \overline{\Omega'' \setminus \bigcup B(p^i, \varepsilon + \varepsilon')} \) and small \( h \) set
\[ D(z, h) := g^{\varepsilon,\delta} \left( z + \frac{P(z_1)}{P(a_1)} h \right) \]
and
\[ v(z, h) = D(z, h) + D(z, -h) + \frac{C_6}{|P(a_1)|^2} (|z - p^i|^2 - R^2) |h|^2, \]
so that
\[ D(z, 0) = g^{\varepsilon,\delta}(z), \]
\[ D(a, h) = g^{\varepsilon,\delta}(a + h), \]
$v$ is psh in $z$, and

$$v(a, h) \geq g^{\varepsilon, \delta}(a + h) + g^{\varepsilon, \delta}(a - h) - \frac{C_6 R^2}{|P(a_1)|^2} |h|^2. \quad (4.9)$$

If $C_6$ is sufficiently big and $h$ sufficiently small, then

$$(Mv(\cdot, h))^{1/n} \geq \left( 1 + \frac{P'(z_1) h_1}{P(a_1)} \right)^{2/n} + \left( 1 - \frac{P'(z_1) h_1}{P(a_1)} \right)^{2/n} \delta^{1/n} + \frac{C_6}{|P(a_1)|^2} |h|^2 \geq (2\delta)^{1/n}.$$  

The Taylor expansion of $D(z, \cdot)$ about the origin gives

$$v(z, h) \leq D(z, h) + D(z, -h) \leq 2g^{\varepsilon, \delta}(z) + \|\nabla^2(D(z, \cdot))\|_{\hat{B}(0, |h|)} |h|^2.$$  

Since

$$|\nabla^2(D(z, \cdot))\hat{h}| = \frac{|P(z_1)|^2}{|P(a_1)|^2} \left| \nabla^2 g^{\varepsilon, \delta}\left( z + \frac{P(z_1)}{P(a_1)} \hat{h} \right) \right|,$$

we get

$$v(z, h) \leq 2g^{\varepsilon, \delta}(z) + C'' |\hat{h}|^2, \quad z \in \partial\Omega',$$

$$v(z, h) \leq 2g^{\varepsilon, \delta}(z) + C_i' |\hat{h}|^2, \quad z \in \partial B(p't, \varepsilon + \varepsilon'),$$

where

$$C'' = C_7 \|\nabla^2 g^{\varepsilon, \delta}\|_{\Omega \cap \mathbb{T}^n},$$

$$C_i' = C_8 \frac{(\varepsilon + \varepsilon')^2 \|\nabla^2 g^{\varepsilon, \delta}\|_{B(p',t+2\varepsilon') \cap \Omega}}{|P(a_1)|^2}$$

for $h$ small enough. Now we can apply the comparison principle to $v$ and $2g^{\varepsilon, \delta}$. We obtain

$$v(a, h) \leq 2g^{\varepsilon, \delta}(a) + \max\{C', C_1', \ldots, C_k'\} |h|^2.$$  

By (4.8) and (4.9)

$$|\nabla^2 g^{\varepsilon, \delta}(a)| \leq \max\{C', C_1', \ldots, C_k'\} + \frac{C_6 R^2}{|P(a_1)|^2}.$$  

If we let $\Omega' \downarrow \Omega$, $\varepsilon' \downarrow 0$, and use (4.6), (4.7), then the desired estimate follows. \igrant
Regularities of the Pluricomplex Green Function

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