Complex Analysis

Squares of positive \((p, p)\)-forms

**Carrés de \((p, p)\)-formes positives**

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**A R T I C L E  I N F O**

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**A B S T R A C T**

We show that if \(\alpha\) is a positive \((2, 2)\)-form, then so is \(\alpha^2\). We also prove that this is no longer true for forms of higher degree.

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**R É S U M É**

Nous montrons que si \(\alpha\) est une \((2, 2)\)-forme positive alors \(\alpha^2\) l’est aussi. Nous prouvons également que ceci n’est plus vrai pour les formes de degré supérieur.

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1. Introduction

Recall that a \((p, p)\)-form \(\alpha\) in \(\mathbb{C}^n\) is called **positive** (we write \(\alpha \geq 0\)) if for \((1, 0)\)-forms \(\gamma_1, \ldots, \gamma_{n-p}\) one has:

\[
\alpha \wedge i \gamma_1 \wedge \bar{\gamma}_1 \wedge \cdots \wedge i \gamma_{n-p} \wedge \bar{\gamma}_{n-p} \geq 0.
\]

This is a natural geometric condition, positive \((p, p)\)-forms are for example characterized by the following property: for every \(p\)-dimensional subspace \(V\) and a test function \(\varphi \geq 0\), one has:

\[
\int_V \varphi \alpha \geq 0.
\]

It is well known that positive forms are real (that is \(\bar{\alpha} = \alpha\)) and if \(\beta\) is a \((1, 1)\)-form then

\[
\alpha \geq 0, \quad \beta \geq 0 \quad \Rightarrow \quad \alpha \wedge \beta \geq 0.
\]  

(1)

It was shown by Harvey and Knapp [5] (and independently by Bedford and Taylor [1]) that (1) does not hold for all \((p, p)\) and \((q, q)\)-forms \(\alpha\) and \(\beta\), respectively. We refer to Demailly’s book [2], pp. 129–132, for a nice and simple introduction to positive forms.

Dinew [3] gave an explicit example of \((2, 2)\)-forms \(\alpha, \beta\) in \(\mathbb{C}^4\) such that \(\alpha \geq 0, \beta \geq 0\) but \(\alpha \wedge \beta < 0\). We will recall it in the next section. The aim of this note is to show the following, somewhat surprising result:

**Theorem 1.** Assume that \(\alpha\) is a positive \((2, 2)\)-form. Then \(\alpha^2\) is also positive.
It turns out that this phenomenon holds only for $(2, 2)$-forms:

**Theorem 2.** For every $p \geq 3$, there exists a $(p, p)$-form $\alpha$ in $\mathbb{C}^{2p}$ such that $\alpha \succeq 0$ but $\alpha^2 \prec 0$.

We do not know if similar results hold for higher powers of positive forms.

The paper is organized as follows: in Section 2 we present Dinew's criterion for positivity of $(2, 2)$-forms in $\mathbb{C}^4$, which reduces the problem to a certain property of $6 \times 6$ matrices. Further simplification reduces the problem to $4 \times 4$ matrices. We then solve it in Section 3. This is the most technical part of the paper. Higher degree forms are analyzed in Section 4, where a counterpart of Dinew's criterion is showed and Theorem 2 is proved.

2. **Dinew's criterion**

Without loss of generality we may assume that $n = 4$. Let $\omega_1, \ldots, \omega_4$ be a basis of $(\mathbb{C}^4)^*$ such that:

$$dV := i\omega_1 \wedge \bar{\omega}_1 \wedge \cdots \wedge i\omega_4 \wedge \bar{\omega}_4 = \omega_1 \wedge \cdots \wedge \omega_4 \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_4 > 0.$$  

Set

$$\begin{align*}
\Omega_1 &:= \omega_1 \wedge \omega_2, & \Omega_2 &:= \omega_1 \wedge \omega_3, & \Omega_3 &:= \omega_1 \wedge \omega_4, \\
\Omega_4 &:= \omega_2 \wedge \omega_3, & \Omega_5 &:= -\omega_2 \wedge \omega_4, & \Omega_6 &:= \omega_3 \wedge \omega_4.
\end{align*}$$

Then

$$\Omega_j \wedge \Omega_k = \begin{cases} 
\omega_1 \wedge \cdots \wedge \omega_4, & \text{if } k = 7 - j, \\
0, & \text{otherwise}.
\end{cases}$$

With every $(2, 2)$-form $\alpha$ we can associate a $6 \times 6$-matrix $A = (a_{jk})$ by

$$\alpha = \sum_{j,k} a_{jk} \Omega_j \wedge \bar{\Omega}_k.$$  

For

$$\beta = \sum_{j,k} b_{jk} \Omega_j \wedge \bar{\Omega}_k$$

we have:

$$\alpha \wedge \beta = \sum_{j,k} a_{jk} b_{7-j,7-k} dV. \quad (2)$$

The key will be the following criterion from [3]:

**Theorem 3.** $\alpha \succeq 0$ if $zA^Tz \succeq 0$ for all $z \in \mathbb{C}^6$ with $z_1z_6 + z_2z_5 + z_3z_4 = 0$.

**Sketch of proof.** For $\gamma_1 = b_1 \omega_1 + \cdots + b_4 \omega_4$, $\gamma_2 = c_1 \omega_1 + \cdots + c_4 \omega_4$, we have

$$i\gamma_1 \wedge \bar{\gamma}_1 \wedge i\gamma_2 \wedge \bar{\gamma}_2 = \sum_{j,k,l,m=1}^4 b_j \bar{b}_k c_l \bar{c}_m \omega_j \wedge \omega_l \wedge \bar{\omega}_k \wedge \bar{\omega}_m$$

$$= \sum_{j<l} \sum_{k<m} (b_j c_l - b_l c_j)(\bar{b}_k c_m - \bar{c}_k b_m)\omega_j \wedge \omega_l \wedge \bar{\omega}_k \wedge \bar{\omega}_m.$$  

It is now enough to show that the image of the mapping:

$$\mathbb{C}^8 \ni (b_1, \ldots, b_4, c_1, \ldots, c_4)$$

$$\mapsto (b_1 \bar{c}_2 - b_2 \bar{c}_1, b_1 \bar{c}_3 - b_3 \bar{c}_1, b_1 \bar{c}_4 - b_4 \bar{c}_1, b_2 \bar{c}_3 - b_3 \bar{c}_2, b_2 \bar{c}_4 - b_4 \bar{c}_2, b_3 \bar{c}_4 - b_4 \bar{c}_3) \in \mathbb{C}^6$$

is precisely $\{z \in \mathbb{C}^6 : z_1z_6 + z_2z_5 + z_3z_4 = 0\}$. Indeed, it is a well-known fact that the image of the Plücker embedding of the 4-dimensional Grassmannian $G(2,4)$ in $P(A^2\mathbb{C}^4) \cong \mathbb{P}^5$ is the quadric defined by the above equation. \( \square \)

Using Theorem 3 and an idea from [3], we can show:
Proposition 4. The form
\[ \alpha_a = \sum_{j=1}^{6} \Omega_j \wedge \bar{\Omega}_j + a \Omega_1 \wedge \bar{\Omega}_6 + b \Omega_6 \wedge \bar{\Omega}_1 \]
is positive if and only if \(|a| \leq 2\).

Proof. We have:
\[ \ddot{z}Az^T = |z|^2 + 2 \text{Re}(a \bar{z}_1 z_6) \geq 2|z_1 z_6| + 2|z_2 z_5 + z_3 z_4| + 2 \text{Re}(a \bar{z}_1 z_6). \]
If \(z_1 z_6 + z_2 z_5 + z_3 z_4 = 0\) and \(|a| \leq 2\) we clearly get \(\ddot{z}Az^T \geq 0\). If we take \(z_1, z_6\) with \(\bar{z}_1 z_6 = -a\), \(|z_1| = |z_6|\) and \(z_2, \ldots, z_5\) with \(z_2 z_5 + z_3 z_4 = -z_1 z_6\) then \(\ddot{z}Az^T = 2|a|(2 - |a|). \)

By (2):
\[ \alpha_a \wedge \alpha_b = 2(3 + \text{Re}(\bar{a}b)) \, dV. \]
Therefore, \(\alpha_2, \alpha_{-2}\) are positive, but \(\alpha_2 \wedge \alpha_{-2} < 0\).

In view of Theorem 3, we see that Theorem 1 is equivalent to the following:

Theorem 5. Let \(A = (a_{jk}) \in \mathbb{C}^{6 \times 6}\) be hermitian and such that \(\ddot{z}Az^T \geq 0\) for \(z \in \mathbb{C}^6\) with \(z_1 z_6 + z_2 z_5 + z_3 z_4 = 0\). Then
\[ \sum_{j,k=1}^{6} a_{jk}a_{7-j,7-k} \geq 0. \]

We will need the following technical reduction:

Lemma 6. For every \((2,2)\)-form \(\alpha\) in \(\mathbb{C}^4\), we can find a basis \(\omega_1, \ldots, \omega_4\) of \((\mathbb{C}^4)^*\) such that:
\[ \alpha \wedge \omega_1 \wedge \omega_2 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2 = \alpha \wedge \omega_1 \wedge \omega_2 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2 = 0 \]
for \(j = 3, 4\).

Proof. We may assume that \(\alpha \neq 0\), then we can find \(\omega_1, \omega_2 \in (\mathbb{C}^4)^*\) such that
\[ \alpha \wedge \omega_1 \wedge \omega_2 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2 = \alpha \wedge \omega_1 \wedge \omega_2 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2 = 0. \]

By \(V_1\) denote the subspace spanned by \(\omega_1, \omega_2\) and by \(V_2\) the subspace of all \(\omega \in (\mathbb{C}^4)^*\) satisfying (3) with \(\omega_j\) replaced by \(\omega\). Then \(\dim V_1 = 2, \dim V_2 \geq 2\), and by (4) we infer \(V_1 \cap V_2 = \{0\}\), hence \((\mathbb{C}^4)^* = V_1 \oplus V_2\).

3. Proof of Theorem 5

By Lemma 6 we may assume that the matrix from Theorem 5 satisfies
\[ a_{26} = a_{36} = a_{46} = a_{56} = 0 \]
and
\[ a_{62} = a_{63} = a_{64} = a_{65} = 0. \]

Then
\[ \sum_{j,k=1}^{6} a_{jk}a_{7-j,7-k} = \sum_{j,k=2}^{5} a_{jk}a_{7-j,7-k} + 2(a_{11}a_{66} + |a_{16}|^2). \]

Therefore Theorem 5 is in fact equivalent to the following result:

Theorem 7. Let \(A = (a_{jk}) \in \mathbb{C}^{4 \times 4}\) be hermitian and such that \(\ddot{z}Az^T \geq 0\) for \(z \in \mathbb{C}^4\) with \(z_1 z_4 + z_2 z_3 = 0\). Then
\[ \sum_{j,k=1}^{4} a_{jk}a_{5-j,5-k} \geq 0. \]
We can then compute
\[ \tilde{z}A^T = \lambda_1 + 2 \text{Re}(a\zeta) + \lambda_2 |\zeta|^2 + 2 \text{Re}\left[ (b - a\zeta + \beta \bar{\zeta} + c|\zeta|^2)w \right] + (\lambda_3 + 2 \text{Re}(d\zeta) + \lambda_4 |\zeta|^2)|w|^2. \]
Therefore \( A \) satisfies the assumption if and only if for every \( \zeta \in \mathbb{C} \)
\[ |a| \leq \sqrt{\lambda_1 \lambda_2}, \quad |b| \leq \sqrt{\lambda_1 \lambda_3}, \quad |c| \leq \sqrt{\lambda_2 \lambda_4}, \quad |d| \leq \sqrt{\lambda_3 \lambda_4}. \] (6)
and for every \( \zeta \in \mathbb{C} \)
\[ |b - a\zeta + \beta \bar{\zeta} + c|\zeta|^2|^2 \leq (\lambda_1 + 2 \text{Re}(a\zeta) + \lambda_2 |\zeta|^2)(\lambda_3 + 2 \text{Re}(d\zeta) + \lambda_4 |\zeta|^2). \] (7)
In our case (5) is equivalent to
\[ 4 \text{Re}(a\bar{d} + b\bar{c}) \leq 2(\lambda_1 \lambda_4 + \lambda_2 \lambda_3) + 2(|\alpha|^2 + |\beta|^2). \]
We will in fact prove something more:
\[ 4 \text{Re}(a\bar{d} + b\bar{c}) \leq (\sqrt{\lambda_1 \lambda_4} + \sqrt{\lambda_2 \lambda_3})^2 + (|\alpha| + |\beta|)^2. \] (8)
Without loss of generality, we may assume that:
\[ \text{Re}(a\bar{d}) > 0, \quad \text{Re}(b\bar{c}) > 0, \]
for if for example \( \text{Re}(a\bar{d}) \leq 0 \) then by (6)
\[ 4 \text{Re}(a\bar{d} + b\bar{c}) \leq 4 \text{Re}(b\bar{c}) \leq 4\sqrt{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \leq (\sqrt{\lambda_1 \lambda_4} + \sqrt{\lambda_2 \lambda_3})^2. \]
Set \( u := \text{Re}(a\bar{d}) \) and \( \zeta := -\bar{d}/|d| \), where \( r > 0 \) will be determined later. Then we can write the right-hand side of (7) as follows:
\[ \left( \lambda_1 - \frac{2ur}{|d|} + \lambda_2 r^2 \right)(\lambda_3 - 2r|d| + \lambda_4 r^2) \]
\[ = (\lambda_1 + \lambda_2 r^2)(\lambda_3 + \lambda_4 r^2) + 4ur^2 - 2r \left[ \lambda_1 |d| + \lambda_3 \frac{u}{|d|} + r^2 \left( \lambda_2 |d| + \lambda_4 \frac{u}{|d|} \right) \right] \]
\[ \leq (\lambda_1 + \lambda_2 r^2)(\lambda_3 + \lambda_4 r^2) + 4ur^2 - 2r(\sqrt{\lambda_1 \lambda_4} + \sqrt{\lambda_2 \lambda_3}) \sqrt{u} \]
\[ = (\sqrt{\lambda_1 \lambda_4} + \sqrt{\lambda_2 \lambda_3} - 2\sqrt{u})^2 r^2 + (\sqrt{\lambda_1 \lambda_3} - \sqrt{\lambda_2 \lambda_4})^2. \]
For \( r = (\frac{\lambda_1 \lambda_3}{\lambda_2 \lambda_4})^{1/4} \) from (7) we thus obtain:
\[ \left| \frac{b}{r} + \frac{\bar{d}}{|d|} \alpha - \frac{d}{|d|} \beta + cr \right| \leq \sqrt{\lambda_1 \lambda_4} + \sqrt{\lambda_2 \lambda_3} - 2\sqrt{u}. \]
We also have:
\[ \left| \frac{b}{r} + \frac{\bar{d}}{|d|} (\alpha - \beta) + cr \right| \geq \left| \frac{b}{r} + cr \right| - |\alpha| - |\beta| \geq 2\sqrt{\text{Re}(b\bar{c})} - |\alpha| - |\beta| \]
and therefore:
\[ 2\sqrt{\text{Re}(a\bar{d})} + 2\sqrt{\text{Re}(b\bar{c})} \leq \sqrt{\lambda_1 \lambda_4} + \sqrt{\lambda_2 \lambda_3} + |\alpha| + |\beta|. \]
To get (8), we can now use the following fact: if \( 0 \leq a_1 \leq x, 0 \leq a_2 \leq x \) and \( a_1 + a_2 \leq x + y \) then \( a_1^2 + a_2^2 \leq x^2 + y^2 \). This can be easily verified: if \( a_1 + a_2 \leq x \) then \( a_1^2 + a_2^2 \leq x^2 \) and if \( a_1 + a_2 \geq x \) then
\[ x^2 + y^2 \geq x^2 + (a_1 + a_2 - x)^2 = a_1^2 + a_2^2 + 2x(x - a_1)(x - a_2). \]
4. (p, p)-Forms in $\mathbb{C}^{2p}$

In $\mathbb{C}^{2p}$ we choose the positive volume form:

$$dV := i\, dz_1 \wedge dz_2 \wedge \cdots \wedge i\, dz_{2p} \wedge d\bar{z}_{2p} = dz_1 \wedge \cdots \wedge dz_{2p} \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{2p}.$$ 

By $\mathcal{I}$ we will denote the set of subscripts $J = (j_1, \ldots, j_p)$ such that $1 \leq j_1 < \cdots < j_p \leq 2p$. For every $J \in \mathcal{I}$ there exists unique $J' \in \mathcal{I}$ such that $J \cup J' = \{1, \ldots, 2p\}$. We also denote $dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ and $\varepsilon_J = \pm 1$ is defined in such a way that:

$$dz_J \wedge dz_{J'} = \varepsilon_J \, dz_1 \wedge \cdots \wedge dz_{2p}.$$ 

Note that:

$$\varepsilon_J \varepsilon_{J'} = (-1)^p.$$

(9)

With every $(p, p)$-form $\alpha$ in $\mathbb{C}^{2p}$ we can associate an $N \times N$-matrix $(a_{JK})$, where $N = \# \mathcal{I} = \binom{2p}{p}! / (p!)^2$

by

$$\alpha = \sum_{J,K} a_{JK} i\, dz_{j_1} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge i\, dz_{j_p} \wedge d\bar{z}_{k_p} = i^p \sum_{J,K} a_{JK} \, dz_J \wedge d\bar{z}_K$$

(10)

(note that $(-1)^{p(p-1)/2} i^p = i^p$). Then

$$\alpha^2 = \sum_{J,K} \varepsilon_J \varepsilon_{K'} a_{JK} a_{J'K'} \, dV$$

(11)

and for $\gamma_1, \ldots, \gamma_p \in (\mathbb{C}^{2p})^*$

$$\alpha \wedge i\gamma_1 \wedge \cdots \wedge i\gamma_p \wedge \bar{\gamma}_p = \sum_{J,K} a_{JK} \gamma_1 \wedge \cdots \wedge \gamma_p \wedge dz_J \wedge (\bar{\gamma}_1 \wedge \cdots \wedge \gamma_p \wedge d\bar{z}_K).$$

Therefore $(a_{JK})$ has to be positive semi-definite on the image of the Plücker embedding

$$\left((\mathbb{C}^{2p})^p \ni (\gamma_1, \ldots, \gamma_p) \longmapsto \left(\frac{\gamma_1 \wedge \cdots \wedge \gamma_p \wedge dz_J}{dz_1 \wedge \cdots \wedge d\bar{z}_{2p}}\right)_{J \in \mathcal{I}}\right) \in \mathbb{C}^N$$

(12)

which is well known to be a variety in $\mathbb{C}^N$ (see, e.g., [4], p. 64).

We are now ready to prove Theorem 2:

**Proof of Theorem 2.** First note that it is enough to show it for $p = 3$. For if $\alpha$ is a $(3, 3)$-form in $\mathbb{C}^6$ such that $\alpha > 0$ and $\alpha^2 < 0$ then for $p > 3$ we set:

$$\beta := i\, dz_7 \wedge d\bar{z}_7 + \cdots + i\, dz_{2p} \wedge d\bar{z}_{2p}.$$ 

We now have $\alpha \wedge \beta^{p-3} \geq 0$ but $(\alpha \wedge \beta^{p-3})^2 = \alpha^2 \wedge \beta^{2p-6} < 0$.

Set $p = 3$, so that $N = 20$, and order $\mathcal{I} = \{J_1, \ldots, J_{20}\}$ lexicographically. Then the image of the Plücker embedding (12) is in particular contained in the quadric:

$$z_1 z_{20} - z_{10} z_{11} + z_5 z_{16} - z_2 z_{19} = 0.$$ 

(13)

For positive $\alpha, \lambda, \mu$ to be determined later define:

$$\alpha := i\left[\lambda(dz_{j_1} \wedge d\bar{z}_{j_1} + dz_{j_{20}} \wedge d\bar{z}_{j_{20}}) + \mu \sum_{k \in \{2, 5, 10, 11, 16, 19\}} dz_{j_k} \wedge d\bar{z}_{j_k} + a(dz_{j_1} \wedge d\bar{z}_{j_{20}} + dz_{j_{20}} \wedge d\bar{z}_{j_1})\right].$$ 

Then, similarly as in the proof of Proposition 4,

$$2A z^T = \lambda (|z_1|^2 + |z_{20}|^2) + \mu (|z_2|^2 + |z_5|^2 + |z_{10}|^2 + |z_{16}|^2 + |z_{11}|^2 + |z_{19}|^2) + 2a \text{Re}(\bar{z}_1 z_{20})$$

$$\geq 2(\lambda - a)|z_1 z_{20}| + 2|\mu| - z_{10} z_{11} + z_5 z_{16} - z_2 z_{19}$$

$$= 2(\lambda + \mu - a)|z_1 z_{20}|$$ 

if $z$ satisfies (13). Therefore $\alpha \geq 0$ if $a \leq \lambda + \mu$. 

On the other hand, by (11) and (9):
\[ \alpha^2 = 2(\lambda^2 + 3\mu^2 - a^2) dV. \]
We see that if we take \( a = \lambda + \mu \) and \( \lambda > \mu > 0 \), then \( \alpha \geq 0 \) but \( \alpha^2 < 0 \).

References