ON GEODESICS IN THE SPACE OF KÄHLER METRICS

Zbigniew Błocki

Dedicated to Professor Shing-Tung Yau on the occasion of his 60th birthday

Abstract. We show that geodesics in the space of Kähler metrics are of class $C^{1,1}$, provided that the manifold has nonnegative bisectional curvature. X.X.Chen has proved that these geodesics have bounded mixed complex derivatives (without any curvature assumption). We also analyze his proof of this, and of the fact that the space of Kähler metrics with the distance defined by its Riemannian structure is a metric space.

1. Introduction

Let M be a compact complex manifold of dimension n with Kähler form ω. We consider the space of Kähler potentials

$$\mathcal{H} := \{\varphi \in C^\infty(M) : \omega_\varphi := \omega + dd^c\varphi > 0\}$$

and the Kähler class

$$\mathcal{K} := \{\omega_\varphi : \varphi \in \mathcal{H}\}.$$

We can treat $\mathcal{H}$ as an open subset in the space $C^\infty(M)$ with topology of uniform convergence of all partial derivatives and differential structure defined by $C^\infty(U, M) := C^\infty(M \times U)$, for any region $U \subset \mathbb{R}^m$.

For $\varphi \in \mathcal{H}$ we can associate the tangent space $T_\varphi\mathcal{H}$ with $C^\infty(M)$. Mabuchi [19] introduced a Riemannian structure on $\mathcal{H}$ as follows:

$$\langle \psi, \eta \rangle := \int_M \psi\eta\omega_\varphi^n, \quad \psi, \eta \in T_\varphi\mathcal{H}.$$

One can check that it gives the following Levi-Civita connection: for a smooth curve $\varphi \in C^\infty([a,b], \mathcal{H})$ (which we treat as an element of $C^\infty(M \times [a,b])$) and ψ, a smooth vector field on φ (which we also treat as an element of $C^\infty(M \times [a,b])$), we have

$$\nabla_{\dot\varphi}\psi = \ddot\psi - \frac{1}{2}\langle \nabla\dot\psi, \nabla\dot\varphi \rangle,$$

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where $\dot\varphi = \partial\varphi/\partial t$ and ∇, $\langle\cdot,\cdot\rangle$ are taken w.r.t. the metric ω_φ. This connection is compatible with the Riemannian structure: if η is another vector field on φ then

$$\frac{d}{dt}\langle\psi,\eta\rangle = \langle\nabla_{\dot\varphi}\psi,\eta\rangle + \langle\psi,\nabla_{\dot\varphi}\eta\rangle.$$

A curve φ is a geodesic if

$$\ddot\varphi - \frac{1}{2}|\nabla\dot\varphi|^2 = 0 \tag{1.1}$$

(with ∇ and $|\cdot|$ taken w.r.t. ω_φ).

This Riemannian structure on $\mathcal{H}$ also gives a structure on $\mathcal{K}$ which is independent of the choice of ω. It is defined by the restriction to the subspace $\mathcal{H}_0 := I^{-1}(0)$, where the functional

$$I(\varphi) := \frac{1}{V}\sum_{p=0}^{n}\frac{1}{p+1}\int_M \varphi\,\omega_\varphi^p\wedge\omega^{n-p}$$

($V = \int_M \omega^n$) is obtained from the splitting of the tangent space

$$T_\varphi\mathcal{H} = \mathbb{R}\oplus\{\psi\in C^\infty(M) : \int_M \psi\omega_\varphi^n = 0\}.$$

(I is sometimes called the Aubin-Yau functional.) One can easily show that if φ is a geodesic in $\mathcal{H}$ then $\varphi - I(\varphi)$ is a geodesic in $\mathcal{H}_0$, and thus for example $\mathcal{H}$ being geodesically convex would imply that so is $\mathcal{K}$.

For an arbitrary curve $\varphi\in C^\infty([a,b],\mathcal{H})$ we define its length in a standard way:

$$l(\varphi) := \int_a^b |\dot\varphi|dt = \int_a^b \sqrt{\int_M \dot\varphi^2\omega_\varphi^n}\,dt$$

and for $\varphi_0,\varphi_1\in\mathcal{H}$ their *distance* by

$$d(\varphi_0,\varphi_1) := \inf\{l(\varphi) : \varphi\in C^\infty([0,1],\mathcal{H}), \varphi(0)=\varphi_0, \varphi(1)=\varphi_1\}$$

(every pair $\varphi_0,\varphi_1\in\mathcal{H}$ can be connected by the curve $(1-t)\varphi_0 + t\varphi_1$ in $\mathcal{H}$).

The following result was proved in [8]:

Theorem 1.1. $(\mathcal{H}, d)$ *is a metric space.*

The only problem with this theorem is whether $d(\varphi_0,\varphi_1) > 0$ for $\varphi_0\neq\varphi_1$. We will show the following quantitative version of this statement (which is a slight improvement of related estimates from [10] and [8]):

Theorem 1.2. *For $\varphi_0,\varphi_1\in\mathcal{H}$ we have*

$$d(\varphi_0,\varphi_1) \geq \sqrt{\max\Big\{\int_{\{\varphi_0>\varphi_1\}}(\varphi_0-\varphi_1)^2\omega_{\varphi_0}^n, \int_{\{\varphi_1>\varphi_0\}}(\varphi_1-\varphi_0)^2\omega_{\varphi_1}^n\Big\}}.$$

The proof of this result consists of two steps: establishing such an estimate for a geodesic distance and then showing that d is equal to the geodesic distance. Very existence of a geodesic connecting two arbitrary points in $\mathcal{H}$ is still an open problem. It was observed by Semmes [22] and Donaldson [10] that the equation (1.1), after the change of variables $t = \log|\zeta|$, $\zeta \in \mathbb{C}_*$, is equivalent to the homogeneous complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^{n+1} = 0.$$

To find a geodesic connecting $\varphi_0, \varphi_1 \in \mathcal{H}$ one thus have to find a solution to the following Dirichlet problem:

$$\begin{cases} \varphi \in C^\infty(M \times \{e^0 \le |\zeta| \le e^1\}), \\ \omega + dd^c\varphi(\cdot,\zeta) > 0, \quad e^0 \le |\zeta| \le e^1, \\ (\omega + dd^c\varphi)^{n+1} = 0, \\ \varphi(\cdot,\zeta) = \varphi_j, \quad |\zeta| = e^j,\ j = 0,1. \end{cases} \tag{1.2}$$

More generally, we may consider the following problem: for a compact Kähler manifold $\widetilde{M}$ (of dimension N) with smooth boundary, a Kähler form $\widetilde{\omega}$ on $\widetilde{M}$, and $\psi \in C^\infty(\partial M)$, we look for φ satisfying

$$\begin{cases} \varphi \in C^\infty(\widetilde{M}) \\ \widetilde{\omega} + dd^c\varphi \ge 0, \\ (\widetilde{\omega} + dd^c\varphi)^N = 0, \\ \varphi = \psi, \quad \text{on } \partial\widetilde{M}. \end{cases} \tag{1.3}$$

(In case of problem (1.2) we take the Kähler form $\widetilde{\omega} := \omega + dd^c|\zeta|^2$ and subtract $|\zeta|^2$ from the obtained solution.) Uniqueness (even among solutions that are much less regular than C^∞) of (1.3) is a direct consequence of a comparison principle for the complex Monge-Ampère equation (see Proposition 2.2 below). However, the following example of Gamelin and Sibony [11] shows that in general a $C^{1,1}$-regularity of the solution is the best one can hope for: the function

$$u(z,w) := (\max\{0, |z|^2 - \frac{1}{2}, |w|^2 - \frac{1}{2}\})^2$$

is C^∞-smooth on the boundary of the unit ball of $\mathbb{C}^2$, satisfies $(u_{j\bar{k}}) \ge 0$ (where we use the notation $u_j = \partial u/\partial z^j$, $u_{\bar{k}} = \partial u/\partial \bar{z}^k$), $\det(u_{j\bar{k}}) = 0$, but it is not C^2. On the other hand, as shown in [14], in case of toric manifolds $\mathcal{H}$ (and thus $\mathcal{K}$) is geodesically convex (or equivalently, (1.2) can be solved for such manifolds). The general case remains open.

Chen [8] showed that any $\varphi_0, \varphi_1 \in \mathcal{H}$ can be joined by a $C^{1,1}$-geodesic. What he was really proving was that for any $\varphi_0, \varphi_1 \in \mathcal{H}$ there exists a weak geodesic connecting φ_0, φ_1 whose Laplacian is bounded (or equivalently, its mixed complex derivatives are bounded; by a weak geodesic we mean that it satisfies $\omega_\varphi \ge 0$ instead of $\omega_\varphi > 0$ and that the intermediate metrics might be less regular than C^∞). We will slightly improve this result if the bisectional curvature is nonnegative: we will show that this geodesic is then really of class $C^{1,1}$ (that is all partial derivatives of second order are bounded).

The key step (also in the proof of Theorems 1.1 and 1.2) is considering approximations of geodesics which will lead to a nondegenerate Monge-Ampère equation: for $\varepsilon > 0$ a curve $\varphi \in C^\infty([a,b],\mathcal{H})$ is called an ε-geodesic if

$$(\ddot{\varphi} - \frac{1}{2}|\nabla\dot{\varphi}|^2)\omega_\varphi^n = \varepsilon\omega^n. \tag{1.1'}$$

This leads to the following modification of (1.2):

$$\text{(1.2')}\qquad \begin{cases} \varphi\in C^\infty(M\times\{e^0\le|\zeta|\le e^1\}),\\ \omega+dd^c\varphi(\cdot,\zeta)>0,\quad e^0\le|\zeta|\le e^1,\\ (\omega+dd^c\varphi)^{n+1}=4\varepsilon|\zeta|^2(\omega+dd^c|\zeta|^2)^{n+1},\\ \varphi(\cdot,\zeta)=\varphi_j,\quad |\zeta|=e^j,\ j=0,1. \end{cases}$$

More generally, for positive $f\in C^\infty(\widetilde{M})$ we get the following modification of (1.3):

$$\text{(1.3')}\qquad \begin{cases} \varphi\in C^\infty(\widetilde{M})\\ \widetilde{\omega}+dd^c\varphi>0,\\ (\widetilde{\omega}+dd^c\varphi)^N=f\widetilde{\omega}^N,\\ \varphi=\psi,\quad \text{on } \partial\widetilde{M}. \end{cases}$$

We will sketch how to prove the following result (see also [21], p. 68):

Theorem 1.3. *If $(\widetilde{M},\widetilde{\omega})$ is a compact Kähler manifold with smooth nonempty boundary, $\psi\in C^\infty(\widetilde{M})$ is such that $\widetilde{\omega}+dd^c\psi>0$, $(\widetilde{\omega}+dd^c\psi)^N\ge f\widetilde{\omega}^N$, and $f\in C^\infty(\widetilde{M})$, $f>0$, then (1.3') has a unique solution.*

For bounded strongly pseudoconvex domains in $\mathbb{C}^N$ this was proved in [7], and in [13] without the assumption of strict pseudoconvexity. Many of the estimates from these papers carry on without much change to our situation, but there are two major exceptions: interior gradient and interior $C^{1,1}$-estimates. As for the gradient estimate, for the proof of Theorem 1.3 one can either use the blowing-up analysis from [8] or estimates from [6] and [15] (proved independently from each other and also from [17], where such a gradient estimate is proved in the non-degenerate case). Concerning the $C^{1,1}$-estimate, we want to stress that it is necessary to establish it in order to prove existence of solutions of the non-degenerate equation (1.3') (and only then can one obtain solutions of the degenerate equation (1.3) by approximation). Our way to deal with this is Theorem 3.4 below (which is the main result of the paper). In an analogous situation a similar estimate is given in Lemma 3.12 from [9] which is proved by a different method, using more standard elliptic theory (we are grateful to X.X. Chen for calling our attention to this reference).

As a result we obtain the following regularity of the degenerate equation for manifolds with flat boundary (we say that $\partial\widetilde{M}$ is *flat* if for every $w\in\partial\widetilde{M}$ there exists a holomorphic change of variables near w such that $\partial\widetilde{M}$ there is of the form $\{\operatorname{Re} z^N=0\}$) (which in addition implies the aforementioned regularity of geodesics in the nonnegative bisectional curvature case):

Theorem 1.4. *Assume that $\partial\widetilde{M}$ is flat and let $\psi\in C^{3,1}(\widetilde{M})$ be such that $\widetilde{\omega}_\psi>0$. Then there exists unique φ with bounded Laplacian in $\widetilde{M}$ (so in particular $\varphi\in C^{1,\alpha}(\operatorname{int}\widetilde{M})$ for $\alpha<1$) such that $\widetilde{\omega}_\varphi\ge0$, $\widetilde{\omega}_\varphi^N=0$ and $\varphi=\psi$ on $\partial\widetilde{M}$. If in particular $(\widetilde{M},\widetilde{\omega})$ has nonnegative bisectional curvature then $\varphi\in C^{1,1}(\widetilde{M})$.*

The organization of the paper is as follows: in Section 2 we prove a general comparison principle for the degenerate Monge-Ampère equation on compact Kähler manifolds with boundary (Theorem 2.3). To prove the most general result we have to follow a method from [5]. In Section 3 we discuss a priori estimates for the Monge-Ampère equation, sketch the proof of Theorem 1.3 and give a proof of Theorem 1.4. In Section 4 we discuss Chen's proof of Theorem 1.1. One thing which should be analyzed a bit more carefully is the proof of Theorem 5 in [8] on p. 219: it is implicitely assumed there that E is bounded away from 0, which is only the case when the curve does not intersect the origin. In the proof of Theorem 4.4 (and in Lemma 4.5) we show how to deal with this extra assumption.

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2. A general comparison principle

Let $(\widetilde{M}, \widetilde{\omega})$ be a compact Kähler manifold with nonempty boundary $\partial\widetilde{M}$ (not necessarily smooth). We start with two simple special cases covering in particular uniqueness in problems (1.3') and (1.3).

Proposition 2.1. *Assume that $\varphi_1, \varphi_2 \in C^2(\widetilde{M})$ are such that $\widetilde{\omega}_{\varphi_1} > 0$, $\widetilde{\omega}_{\varphi_2} > 0$, $\widetilde{\omega}_{\varphi_1}^N \geq \widetilde{\omega}_{\varphi_2}^N$, and $\varphi_1 \leq \varphi_2$ on $\partial\widetilde{M}$. Then $\varphi_1 \leq \varphi_2$ on $\widetilde{M}$.*

Proof. We have

$$0 \leq \widetilde{\omega}_{\varphi_1}^N - \widetilde{\omega}_{\varphi_2}^N = dd^c(\varphi_1 - \varphi_2) \wedge T,$$

where

$$T = \sum_{j=0}^{N-1} \widetilde{\omega}_{\varphi_1}^j \wedge \widetilde{\omega}_{\varphi_2}^{N-1-j} > 0.$$

It follows that $\varphi_1 - \varphi_2$ satisfies the maximum principle. $\square$

Concerning the problem (1.3), one can show uniqueness even among generalized solutions in the sense of Bedford and Taylor [3], [4]:

Proposition 2.2. *Let φ_j be continuous on $\widetilde{M}$ and such that $\widetilde{\omega}_{\varphi_j} \geq 0$, $j = 1, 2$. Assume that $\widetilde{\omega}_{\varphi_2}^N = 0$ and $\varphi_1 \leq \varphi_2$ on $\partial\widetilde{M}$. Then $\varphi_1 \leq \varphi_2$ on $\widetilde{M}$.*

Proof. (See also [20], p. 144). Without loss of generality we may assume that φ_1, φ_2 are nonnegative. Suppose $\{\varphi_1 > \varphi_2\} \neq \emptyset$. Then $S := \{\lambda\varphi_1 > \varphi_2\} \neq \emptyset$ for some $\lambda < 1$. For $\varepsilon > 0$ set $\varphi_\varepsilon := \max\{\lambda\varphi_1, \varphi_2 + \varepsilon\}$. Then φ_ε decreases to $\lambda\varphi_1$ on S as ε decreases to 0, but $\varphi_\varepsilon = \varphi_2 + \varepsilon$ near ∂S. It follows that

$$\int_S \widetilde{\omega}_{\varphi_2}^N = \liminf_{\varepsilon \to 0} \int_S \widetilde{\omega}_{\varphi_\varepsilon}^N \geq \int_S \widetilde{\omega}_{\lambda\varphi_1}^N > \lambda^N \int_S \widetilde{\omega}_{\varphi_1}^N$$

which contradicts the fact that $\widetilde{\omega}_{\varphi_2}^N = 0$. $\square$

The following result covers both Propositions 2.1 and 2.2 but is a bit more difficult to prove, we follow a method from [5]:

Theorem 2.3. *Let φ_j be continuous on $\widetilde{M}$ and such that $\widetilde{\omega}_{\varphi_j} \geq 0$, $j = 1, 2$. Assume that $\widetilde{\omega}_{\varphi_2}^N \leq \widetilde{\omega}_{\varphi_1}^N$ and $\varphi_1 \leq \varphi_2$ on $\partial\widetilde{M}$. Then $\varphi_1 \leq \varphi_2$ on $\widetilde{M}$.*

Proof. For $\varepsilon > 0$ let $\varphi_\varepsilon := \max\{\varphi_1, \varphi_2 + \varepsilon\}$. Then $\varphi_\varepsilon = \varphi_2 + \varepsilon$ near $\partial\widetilde{M}$. Therefore, by the well known inequality for continuous plurisubharmonic functions:

$$(dd^c \max\{u, v\})^N \geq \chi_{\{u \geq v\}}(dd^c u)^N + \chi_{\{u<v\}}(dd^c v)^N,$$

it follows that without loss of generality we may assume that $\varphi_1 = \varphi_2$ near $\partial\widetilde{M}$ and $\varphi_1 \geq \varphi_2$ on $\widetilde{M}$. We have to show that $\varphi_1 = \varphi_2$.

For simplicity of presentation, from now on we will assume that $N = 2$, the general case is similar (see [5] for necessary modifications). We have

$$0 \leq \widetilde{\omega}_{\varphi_1}^2 - \widetilde{\omega}_{\varphi_2}^2 = dd^c\rho \wedge (\widetilde{\omega}_{\varphi_1} + \widetilde{\omega}_{\varphi_2}),$$

where $\rho := \varphi_1 - \varphi_2$. Integrating by parts

$$0 \leq \int_{\widetilde{M}} \rho\, dd^c\rho \wedge (\widetilde{\omega}_{\varphi_1} + \widetilde{\omega}_{\varphi_2}) = -\int_{\widetilde{M}} d\rho \wedge d^c\rho \wedge (\widetilde{\omega}_{\varphi_1} + \widetilde{\omega}_{\varphi_2}),$$

and thus

$$d\rho \wedge d^c\rho \wedge \widetilde{\omega}_{\varphi_1} = d\rho \wedge d^c\rho \wedge \widetilde{\omega}_{\varphi_2} = 0. \tag{2.1}$$

It will be enough to show that

$$d\rho \wedge d^c\rho \wedge \widetilde{\omega} = 0. \tag{2.2}$$

We have

$$\begin{aligned}\int_{\widetilde{M}} d\rho \wedge d^c\rho \wedge dd^c\varphi_1 &= -\int_{\widetilde{M}} \rho\, dd^c\rho \wedge dd^c\varphi_1 \\ &= \int_{\widetilde{M}} d\rho \wedge d^c\varphi_1 \wedge dd^c\rho = \int_{\widetilde{M}} d\rho \wedge d^c\varphi_1 \wedge (\widetilde{\omega}_{\varphi_1} - \widetilde{\omega}_{\varphi_2}).\end{aligned}$$

By the Schwarz inequality

$$\left| \int_{\widetilde{M}} d\rho \wedge d^c\varphi_1 \wedge \widetilde{\omega}_{\varphi_j} \right|^2 \leq \int_{\widetilde{M}} d\rho \wedge d^c\rho \wedge \widetilde{\omega}_{\varphi_j} \int_{\widetilde{M}} d\varphi_1 \wedge d^c\varphi_1 \wedge \widetilde{\omega}_{\varphi_j}.$$

From (2.1) it now follows that

$$\int_{\widetilde{M}} d\rho \wedge d^c\varphi_1 \wedge \widetilde{\omega}_{\varphi_j} = 0$$

and we get (2.2). $\square$

3. A priori estimates

We assume that $(\widetilde{M}, \widetilde{\omega})$ is a compact Kähler manifold with smooth nonempty boundary $\partial\widetilde{M}$. In order to prove Theorem 1.3, by continuity method it is enough to establish for some $\alpha \in (0, 1)$ the a priori estimate

$$||\varphi||_{C^{2,\alpha}(\widetilde{M})} \leq C \tag{3.1}$$

for solutions of (1.3').

Any C^2-solution φ of (1.3') satisfies

$$\psi \le \varphi \le h,$$

where h is the harmonic function on $\widetilde{M}$ with $h = \psi$ on $\partial\widetilde{M}$. It follows that

$$\sup_{\widetilde{M}} |\varphi| \le C,$$

and

$$\sup_{\partial\widetilde{M}} |\nabla\varphi| \le C,$$

where C depends only on $\widetilde{M}$, $\widetilde{\omega}$, and $||\psi||_{C^{0,1}(\widetilde{M})}$.

The next step is an interior gradient estimate. One possibility is the blowing-up analysis of [8] (then one has to consider the $C^{1,1}$-estimate first). A more direct approach is the following estimate from [6] (see also [17] and [15]):

Theorem 3.1. *Let $\varphi \in C^3(\widetilde{M})$ be such that $\widetilde{\omega}_\varphi > 0$ and $\widetilde{\omega}_\varphi^N = f\widetilde{\omega}^N$. Then*

$$|\nabla\varphi| \le C,$$

where C depends only on upper bounds for $|\varphi|$, $\sup_{\partial\widetilde{M}} |\nabla\varphi|$, $||f^{1/N}||_{C^{0,1}(\widetilde{M})}$, N, and a lower bound for the bisectional curvature of $\widetilde{M}$. $\square$

To prove a $C^{1,1}$-estimate on the boundary one has to follow [7] and [13]. One will obtain:

Theorem 3.2. *For $\varphi \in C^4(\widetilde{M})$ with $\widetilde{\omega}_\varphi > 0$ and $\widetilde{\omega}_\varphi^N = f\widetilde{\omega}^N$ one has*

$$\sup_{\partial\widetilde{M}} |\nabla^2\varphi| \le C,$$

where C depends only on $\widetilde{M}$, $\widetilde{\omega}$, on upper bounds for $||\varphi||_{C^{0,1}(\widetilde{M})}$, $||\psi||_{C^{3,1}(\widetilde{M})}$, $||f^{1/N}||_{C^{0,1}(\widetilde{M})}$, and on positive lower bounds for f and for eigenvalues of ω_ψ. $\square$

The proof of Theorem 3.2 is divided into three steps: estimates for tangential-tangential, tangential-normal and normal-normal derivatives. The tangential-tangential case is very simple (see [7]), the tangential-normal one has to follow B. Guan's estimate [13] (this is how it is also done in [8]), and then the normal-normal case one does the same way as in [7].

For the regularity of geodesics in $\mathcal{H}$ one needs to show that the estimate is independent of the lower bound for f. This can be easily done when $\partial\widetilde{M}$ is flat:

Theorem 3.2'. *If $\partial\widetilde{M}$ is flat the estimate in Theorem 3.2 does not depend on a lower bound for f.*

Proof. The estimates for tangential-tangential and tangential-normal derivatives from [7] and [13] do not depend on a lower bound for f, the only problem is the normal-normal case. We work in a ball $B(0, R)$ in $\mathbb{C}^N$ and assume that there int $\widetilde{M}$

is equal to $\{x^N < 0\}$ (we use the notation $z^j = x^j + iy^j$). Write $u = g + \varphi$ and $v = g + \psi$ where $\omega = dd^c g$. We have

$$\det(u_{j\bar{k}}) = \widetilde{f},$$

where $\widetilde{f} = f \det(g_{j\bar{k}})$. On the other hand on $\{x^N = 0\}$

$$\det(u_{j\bar{k}}) = u_{N\bar{N}} \det(u_{j\bar{k}})_{j,k\le N-1} + R = u_{N\bar{N}} \det(v_{j\bar{k}})_{j,k\le N-1} + R,$$

where $|R|$ is under control from above. Therefore at $\{x^N = 0\}$

$$u_{N\bar{N}} \le \frac{C}{\lambda^{N-1}},$$

where $\lambda > 0$ is such that $(v_{j\bar{k}}) \ge \lambda(\delta_{jk})$. □

The normal-normal boundary estimate from the proof of Theorem 3.2' is the only time when we need the assumption of flatness. By Krylov [18] (who used completely different, probabilistic methods), this estimate is however also true without this assumption. Therefore so is Theorem 1.4, even for a more general degenerate, but not necessarily homogeneous, equation. It would be interesting to prove a general normal-normal boundary estimate for the degenerate complex Monge-Ampère equation not using probabilistic methods. In the real case this was done in [16].

We now turn to interior estimates for second derivatives. For the mixed complex derivatives we can use the Aubin-Yau estimate ([1], [24]). We will obtain (see also [6]):

Theorem 3.3. *For $\varphi \in C^4(\widetilde{M})$ with $\widetilde{\omega}_\varphi > 0$ and $\widetilde{\omega}_\varphi^N = f\widetilde{\omega}^N$ one has*

$$\Delta\varphi \le C,$$

where C depends only on upper bounds for $|\varphi|$, $\sup_{\partial\widetilde{M}} \Delta\varphi$, f, N, scalar curvature, and on lower bounds for $f^{1/(N-1)}\Delta(\log f)$ and bisectional curvature. □

The next step is the aforementioned interior $C^{1,1}$-estimate:

Theorem 3.4. *Assume that $\varphi \in C^4(\widetilde{M})$, $\widetilde{\omega}_\varphi > 0$, and $\widetilde{\omega}_\varphi^N = f\widetilde{\omega}^N$. Then*

$$|\nabla^2\varphi| \le C,$$

where C is a constant depending only on upper bounds for N, $|R|$, $|\nabla R|$, $|\varphi|$, $|\nabla\varphi|$, $\Delta\varphi$, $\sup_{\partial\widetilde{M}} |\nabla^2\varphi|$, $||f^{1/N}||_{C^{1,1}(\widetilde{M})}$, $|\nabla(f^{1/2N})|$, and on a lower positive bound for f. If $\widetilde{M}$ has a nonnegative bisectional curvature then the estimate is independent of a lower bound for f.

Remarks. 1. By an upper bound for $|R|$ (resp. $|\nabla R|$) we understand a constant C such that

$$|R(X_1, \dots, X_4)| \le C|X_1| \dots |X_4|, \quad X_1, \dots, X_4 \in T\widetilde{M}$$
$$\text{(resp. } |(\nabla R)(X_1, \dots, X_5)| \le C|X_1| \dots |X_5|, \quad X_1, \dots, X_5 \in T\widetilde{M}).$$

2. Since $h := f^{1/N} \geq 0$, it is an elementary fact that if h extends as a nonnegative $C^{1,1}$ function to some neighborhood of $\widetilde{M}$, then $|\nabla(h^{1/2})|$ is under control, provided that $||h||_{C^{1,1}(\widetilde{M})}$ is (of course depending also on the extension).

Theorem 3.4 in fact can be also used in the proof of the Calabi-Yau theorem (the problem (1.3') with empty boundary) in order to apply the real Evans-Krylov theory directly (and not repeat it in the complex case, see e.g. [23]).

Proof of Theorem 3.4. At a given point of $\widetilde{M}$, let β denote the maximal eigenvalue of the mapping

$$T\widetilde{M} \ni X \longmapsto \nabla_X \nabla\varphi,$$

so that

$$\beta = \max_{X \in T\widetilde{M}\setminus\{0\}} \frac{\langle \nabla_X \nabla\varphi, X\rangle}{|X|^2}.$$

Then β is a continuous function on $\widetilde{M}$ (but not necessarily smooth). It is clear that it is enough to estimate β from above.

Locally we have (using the notation $\partial_j = \partial/\partial z^j$, $\partial_{\bar{j}} = \partial/\partial \bar{z}^j$, and denoting by $(g^{j\bar{k}})$ the inverse of $(g_{j\bar{k}})$, where $\widetilde{\omega} = dd^c g$)

$$\begin{aligned}\nabla_{\partial_j}\nabla\varphi &= \partial_j(g^{p\bar{q}}\varphi_p)\partial_{\bar{q}} + \partial_j(g^{p\bar{q}}\varphi_{\bar{q}})\partial_p + g^{p\bar{q}}\varphi_{\bar{q}}\Gamma^s_{jp}\partial_s \\ &= g^{p\bar{q}}\varphi_{j\bar{q}}\partial_p + (g^{p\bar{q}}\varphi_p)_j\partial_{\bar{q}}.\end{aligned}$$

Therefore for a real vector field $X = X^j\partial_j + \bar{X}^k\partial_{\bar{k}}$

$$\begin{aligned}\langle \nabla_X\nabla\varphi, X\rangle &= 2\mathrm{Re}\, X^j\left(\bar{X}^k\varphi_{j\bar{k}} + X^l g_{l\bar{q}}(g^{p\bar{q}}\varphi_p)_j\right) \\ &= D_X^2\varphi + 2\mathrm{Re}\left(X^jX^l g^{p\bar{q}}g_{j\bar{q}l}\varphi_p\right),\end{aligned}$$

where D_X denotes euclidean directional derivative in direction X.

On $\widetilde{M}$ define

$$\alpha := \beta + |\nabla\varphi|^2 - A\varphi,$$

where A is a positive constant under control that will be specified later. We may assume that α attains maximum at some O in the interior of M (for otherwise we are done). We can find holomorphic coordinates near O such that at O the matrix $(\varphi_{j\bar{k}})$ is diagonal and

$$g_{j\bar{k}} = \delta_{jk}, \quad g_{j\bar{k}l} = g_{j\bar{k}lm} = 0. \tag{3.2}$$

Choose fixed $X = (X^1, \dots, X^N) \in \mathbb{C}^N$ such that at O $|X|^2 (= 2g_{j\bar{k}}X^j\bar{X}^k) = 1$ and

$$\beta = \langle \nabla_X\nabla\varphi, X\rangle.$$

Near O define

$$\widetilde{\beta} := \frac{\langle \nabla_X\nabla\varphi, X\rangle}{|X|^2}$$

and

$$\widetilde{\alpha} := \widetilde{\beta} + |\nabla\varphi|^2 - A\varphi.$$

Then $\widetilde{\beta} \leq \beta$, $\widetilde{\beta}(O) = \beta(O)$, and

$$\widetilde{\alpha} \leq \alpha \leq \alpha(O) = \widetilde{\alpha}(O),$$

so that (locally defined) $\widetilde{\alpha}$ also has a maximum at O, equal to that of α. The advantage of $\widetilde{\alpha}$ though is that it is smooth! It remains to estimate $\widetilde{\beta}(O)$ from above.

Set $u := \varphi + g$, then the Monge-Ampère equation gives near O

$$\det(u_{p\bar{q}}) = \widetilde{f}, \tag{3.3}$$

where

$$\widetilde{f} := f \det(g_{p\bar{q}}).$$

Differentiating (3.3) we will get

$$u^{p\bar{q}} D_X u_{p\bar{q}} = D_X(\log \widetilde{f})$$

and

$$u^{p\bar{q}} D_X^2 u_{p\bar{q}} = D_X^2(\log \widetilde{f}) + u^{p\bar{t}} u^{s\bar{q}} D_X u_{p\bar{q}} D_X u_{s\bar{t}}.$$

At O by (3.2) and (3.3)

$$D_X^2(\log\det(g_{p\bar{q}})) = \sum_p D_X^2 g_{p\bar{p}}$$

and therefore

$$\sum_p \frac{D_X^2 \varphi_{p\bar{p}}}{u_{p\bar{p}}} \geq D_X^2(\log f) + \sum_p D_X^2 g_{p\bar{p}} - \sum_p \frac{D_X^2 g_{p\bar{p}}}{u_{p\bar{p}}}.$$

We also have

$$\frac{1}{N} D_X^2(\log f) = D_X^2(\log f^{1/N}) \geq -\frac{C_1}{f^{1/N}} \geq -C_1 \sum_p \frac{1}{u_{p\bar{p}}}$$

by (3.2) and the inequality between arithmetic and geometric means (where by $C_1, C_2, \dots$ we will denote constants under control). From the fact that $|R|$ is under control we will now get

$$\sum_p \frac{D_X^2 \varphi_{p\bar{p}}}{u_{p\bar{p}}} \geq -C_3 \sum_p \frac{1}{u_{p\bar{p}}} - C_4 \geq -C_5 \sum_p \frac{1}{u_{p\bar{p}}} \tag{3.4}$$

since $u_{p\bar{p}}$ is under control from above.

Using the fact that $|X| = 1$ and $(|X|^2)_p = 0$ at O, combined with (3.2), at O we will get

$$\begin{aligned} \widetilde{\beta}_{p\bar{p}} &= D_X^2 \varphi_{p\bar{p}} + 2\mathrm{Re} \sum_l X^j X^k g_{j\bar{l}k\bar{p}p} \varphi_l \\ &\quad + 2\mathrm{Re} \sum_l X^j X^k g_{j\bar{l}k\bar{p}} \varphi_{lp} - X^j \bar{X}^k g_{j\bar{k}p\bar{p}} D_X^2 \varphi \\ &\geq D_X^2 \varphi_{p\bar{p}} - C_6 - C_7 \sum_l |\varphi_{lp}| - C_8 \widetilde{\beta}, \end{aligned} \tag{3.5}$$

where we used in addition that $|\nabla R|$ is under control.

Near O we have

$$(|\nabla\varphi|^2)_p = (g^{j\bar{k}})_p \varphi_j \varphi_{\bar{k}} + g^{j\bar{k}} \varphi_{jp}\varphi_{\bar{k}} + g^{j\bar{k}}\varphi_j \varphi_{p\bar{k}}.$$

Therefore at O

$$(|\nabla\varphi|^2)_{p\bar{p}} = \sum_{j,k} R_{j\bar{k}p\bar{p}}\varphi_j\varphi_{\bar{k}} + 2\mathrm{Re}\sum_j \varphi_{jp\bar{p}}\varphi_{\bar{j}} + \sum_j |\varphi_{jp}|^2 + \varphi_{p\bar{p}}^2.$$

Since

$$\frac{1}{2}\sum_j |\varphi_{jp}|^2 - C_7 \sum_j |\varphi_{jp}| \geq -C_9,$$

$$2\mathrm{Re}\sum_{j,p} \frac{\varphi_{jp\bar{p}}\varphi_{\bar{j}}}{u_{p\bar{p}}} = 2\mathrm{Re}\sum_j (\log f)_j \varphi_{\bar{j}} \geq -\frac{C_{10}}{f^{1/n}} \geq -C_{10}\sum_p \frac{1}{u_{p\bar{p}}},$$

and

$$\frac{1}{2}\sum_{j,p} \frac{|\varphi_{jp}|^2}{u_{p\bar{p}}} \geq \frac{1}{C_{11}}\widetilde{\beta}^2 - C_{12},$$

from (3.4), (3.5) it follows that

$$\sum_p \frac{\widetilde{\beta}_{p\bar{p}}}{u_{p\bar{p}}} + \sum_p \frac{(|\nabla\varphi|^2)_{p\bar{p}}}{u_{p\bar{p}}} \geq \frac{1}{C_{11}}\widetilde{\beta}^2 - C_{13}\sum_p \frac{1}{u_{p\bar{p}}} - C_8\widetilde{\beta}\sum_p \frac{1}{u_{p\bar{p}}}$$

and finally

$$0 \geq \sum_p \frac{\alpha_{p\bar{p}}}{u_{p\bar{p}}} \geq \frac{1}{C_{11}}\widetilde{\beta}^2 - C_{13}\sum_p \frac{1}{u_{p\bar{p}}} - C_8\widetilde{\beta}\sum_p \frac{1}{u_{p\bar{p}}} + A\sum_p \frac{1}{u_{p\bar{p}}} - AN.$$

If $\widetilde{M}$ has nonnegative bisectional curvature then we may take $C_8 = 0$ (this constant is coming from (3.5)) and $A = C_{13}$. In the general case, if f is controlled from below (which we have not yet used) then $\sum_p 1/u_{p\bar{p}}$ is controlled from above and the theorem follows immediately (with $A = 0$). $\square$

Once we have the $C^{1,1}$-estimate, the $C^{2,\alpha}$-estimate (3.1) in the nondegenerate case follows from [7]. This gives Theorem 1.3 and we also obtain Theorem 1.4 by approximation by smooth solutions.

4. The metric structure of $\mathcal{H}$

The results of Section 2 have the following consequence on ε-geodesics in $\mathcal{H}$:

Theorem 4.1. *For $\varphi_0, \varphi_1 \in \mathcal{H}$ and $\varepsilon > 0$ there exists a unique ε-geodesic φ connecting φ_0 with φ_1. Moreover, it depends smoothly on φ_0, φ_1, i.e. if $\varphi_0, \varphi_1 \in C^\infty([0,1], \mathcal{H})$ then there exists unique $\varphi \in C^\infty([0,1]\times[0,1], \mathcal{H})$ such that $\varphi(0,\cdot) = \varphi_0$, $\varphi(1,\cdot) = \varphi_1$, and $\varphi(\cdot,t)$ is an ε-geodesic for every $t \in [0,1]$.*

In addition, in variables (z,ζ), where $t = \log|\zeta|$, it satisfies the estimate

$$dd^c\varphi \leq C(\omega + dd^c|\zeta|^2),$$

where C is independent of ε (if ε is small). $\square$

(Smooth dependence of solutions of (1.3') on boundary data follows from standard elliptic theory, see e.g. [2], [12].) Our aim is to prove Theorem 1.1. As in [8], we start with the following:

Lemma 4.2. *Let φ be an ε-geodesic connecting $\varphi_0, \varphi_1 \in \mathcal{H}$, and set*

$$E := \int_M \dot\varphi^2 \omega_\varphi^n.$$

Then

$$E \geq \mathcal{E}(\varphi_0, \varphi_1) - 2\varepsilon V ||\dot\varphi||,$$

where

$$\mathcal{E}(\varphi_0, \varphi_1) := \max\Big\{ \int_{\{\varphi_0 > \varphi_1\}} (\varphi_0 - \varphi_1)^2 \omega_{\varphi_0}^n, \int_{\{\varphi_1 > \varphi_0\}} (\varphi_1 - \varphi_0)^2 \omega_{\varphi_1}^n \Big\}$$

and $||\dot\varphi|| = \sup_{M \times [0,1]} |\dot\varphi|$. In particular,

$$l(\varphi)^2 \geq \mathcal{E}(\varphi_0, \varphi_1) - 2\varepsilon V ||\dot\varphi||.$$

Proof. We have

$$\dot E = \int_M (2\dot\varphi\ddot\varphi + \frac{1}{2}\dot\varphi^2 \Delta\dot\varphi)\omega_\varphi^n = 2\int_M \dot\varphi(\ddot\varphi - \frac{1}{2}|\nabla\dot\varphi|^2)\omega_\varphi^n = 2\varepsilon \int_M \dot\varphi \omega^n.$$

Thus $|\dot E| \leq 2\varepsilon V ||\dot\varphi||$ which implies that

$$E(t) \geq \max\{E(0), E(1)\} - 2\varepsilon V ||\dot\varphi||.$$

Since $\ddot\varphi \geq 0$,

$$\dot\varphi(0) \leq \varphi(1) - \varphi(0) \leq \dot\varphi(1).$$

For $z \in M$ with $\varphi_1(z) > \varphi_0(z)$ we thus have $\dot\varphi(z,1)^2 \geq (\varphi_1(z) - \varphi_0(z))^2$. Therefore

$$E(1) \geq \int_{\{\varphi_1 > \varphi_0\}} (\varphi_1 - \varphi_0)^2 \omega_{\varphi_1}^n.$$

Similarly

$$E(0) \geq \int_{\{\varphi_0 > \varphi_1\}} (\varphi_0 - \varphi_1)^2 \omega_{\varphi_0}^n$$

and the desired estimate follows. $\square$

Theorem 4.3. *Suppose $\psi \in C^\infty([0,1], \mathcal{H})$ and $\widetilde\psi \in \mathcal{H} \setminus \psi([0,1])$. For $\varepsilon > 0$ by φ denote an element of $C^\infty([0,1] \times [0,1], \mathcal{H})$ uniquely determined by the following property: $\varphi(\cdot, t)$ is an ε-geodesic connecting $\widetilde\psi$ with $\psi(t)$. Then for ε sufficiently small*

$$l(\varphi(\cdot, 0)) \leq l(\psi) + l(\varphi(\cdot, 1)) + C\varepsilon,$$

where $C > 0$ is independent of ε.

Proof. The proof is essentially the same as that of Theorem 5 in [8]. Note however that it is crucial to assume below that $\widetilde{\psi} \notin \psi([0,1])$. Set

$$l_1(t) := \int_0^t |\dot\psi| d\widetilde{t}, \qquad l_2(t) := l(\varphi(\cdot, t)).$$

It is enough to show that $l_1' + l_2' \geq -C\varepsilon$ on $[0,1]$. We clearly have

$$l_1' = |\dot\psi| = \sqrt{\int_M \dot\psi^2 \omega_\psi^n}.$$

On the other hand,

$$l_2(t) = \int_0^1 \sqrt{E(s,t)}\, ds,$$

where

$$E = \int_M \varphi_s^2 \omega_\varphi^n$$

(using the notation $\varphi_s = \partial\varphi/\partial s$). We have

$$E_s = 2\int_M \varphi_s \nabla_{\varphi_s} \varphi_s \omega_\varphi^n = 2\varepsilon \int_M \varphi_s \omega^n$$

and

$$\begin{aligned} E_t &= \int_M \big(2\varphi_s\varphi_{st} + \frac{1}{2}\varphi_s^2 \Delta\varphi_t\big)\omega_\varphi^n \\ &= 2\int_M \varphi_s\big(\varphi_{st} - \frac{1}{2}\langle \nabla\varphi_s, \nabla\varphi_t\rangle\big)\omega_\varphi^n \\ &= 2\int_M \varphi_s \nabla_{\varphi_s}\varphi_t \omega_\varphi^n \\ &= 2\frac{\partial}{\partial s}\int_M \varphi_s\varphi_t\omega_\varphi^n - 2\int_M \varphi_t \nabla_{\varphi_s}\varphi_s \omega_\varphi^n \\ &= 2\frac{\partial}{\partial s}\int_M \varphi_s\varphi_t\omega_\varphi^n - 2\varepsilon\int_M \varphi_t\omega^n. \end{aligned}$$

Therefore

$$\begin{aligned} l_2' &= \frac{1}{2}\int_0^1 E^{-1/2} E_t\, ds \\ &= \int_0^1 E^{-1/2}\frac{\partial}{\partial s}\int_M \varphi_s\varphi_t\omega_\varphi^n\, ds - \varepsilon\int_0^1 E^{-1/2}\int_M \varphi_t\omega^n\, ds, \end{aligned}$$

and the first term is equal to

$$\left[E^{-1/2}\int_M \varphi_s\varphi_t\omega_\varphi^n\right]_0^1 + \frac{1}{2}\int_0^1 E^{-3/2}E_s\int_M \varphi_s\varphi_t\omega_\varphi^n\, ds.$$

Since $\varphi_t(0,\cdot)=0$, $\varphi_t(1,\cdot)=\dot\varphi$, and

$$E(1,\cdot)=\int_M \eta^2\omega_\varphi^n,$$

where $\eta=\varphi_s(1,\cdot)$, from the Schwarz inequality it follows that

$$l_2'=\left(\int_M \eta^2\omega_\varphi^n\right)^{-1/2}\int_M \eta\dot\varphi\omega_\varphi^n - R \geq -\left(\int_M \dot\varphi^2\omega_\varphi^n\right)^{1/2} - R,$$

where

$$R=\varepsilon\int_0^1 E^{-1/2}\int_M \varphi_t\omega^n\,ds - \varepsilon\int_0^1 E^{-3/2}\int_M \varphi_s\omega^n\int_M \varphi_s\varphi_t\omega_\varphi^n\,ds.$$

By Lemma 4.2

$$E(s,t)\geq \mathcal{E}(\widetilde{\psi},\psi(t))-2\varepsilon V||\varphi_s||.$$

Since $\mathcal{E}(\widetilde{\psi},\psi(t))$ is continuous and positive for $t\in[0,1]$, it follows that for ε sufficiently small

$$E\geq c>0$$

and thus $R\leq C\varepsilon$. $\square$

We are now in position to show that the geodesic distance is the same as d:

Theorem 4.4. *Let φ^ε be an ε-geodesic connecting $\varphi_0,\varphi_1\in\mathcal{H}$. Then*

$$d(\varphi_0,\varphi_1)=\lim_{\varepsilon\to 0^+} l(\varphi^\varepsilon).$$

Proof. Let $\psi\in C^\infty([0,1],\mathcal{H})$ be an arbitrary curve connecting $\varphi_0,\varphi_1\in\mathcal{H}$. We have to show that

$$\overline{\lim_{\varepsilon\to 0^+}}\, l(\varphi^\varepsilon)\leq l(\psi).$$

Without loss of generality we may assume that $\varphi_1\notin\psi([0,1))$. Extend φ^ε to a function from $C^\infty([0,1]\times[0,1),\mathcal{H})$ in such a way that $\varphi^\varepsilon(0,\cdot)\equiv\varphi_1$, $\varphi^\varepsilon(1,\cdot)\equiv\psi$ on $[0,1)$ and $\varphi^\varepsilon(\cdot,t)$ is an ε-geodesic for $t\in[0,1)$. By Theorem 4.3 for $t\in[0,1)$ we have

$$l(\varphi^\varepsilon(\cdot,0))\leq l(\psi|_{[0,t]})+l(\varphi^\varepsilon(\cdot,t))+C(t)\varepsilon.$$

Since clearly

$$\lim_{t\to 1^-} l(\psi|_{[0,t]})=l(\psi),$$

it remains to show that

$$\overline{\lim_{t\to 1^-}}\;\overline{\lim_{\varepsilon\to 0^+}}\, l(\varphi^\varepsilon(\cdot,t))=0.$$

But it follows immediately from the following:

Lemma 4.5. *For an ε-geodesic φ connecting $\varphi_0,\varphi_1\in\mathcal{H}$ we have*

$$l(\varphi)\leq\sqrt{V}\left(||\varphi_0-\varphi_1||_{L^\infty(M)}+\frac{165\,\varepsilon}{\lambda^n}\right),$$

where $\lambda > 0$ is such that $\omega_{\varphi_0} \geq \lambda\omega$, $\omega_{\varphi_1} \geq \lambda\omega$.

Proof. Since $\ddot{\varphi} \geq 0$,

$$\dot{\varphi}(0) \leq \dot{\varphi} \leq \dot{\varphi}(1).$$

So to estimate $|\dot{\varphi}|$ from above we need to bound $\dot{\varphi}(0)$ from below and $\dot{\varphi}(1)$ from above. The function

$$v(\zeta) = b|\zeta|^4 + (a + b - be^4)\log|\zeta| - a - b$$

satisfies $\partial^2 v/\partial\zeta\partial\bar{\zeta} = 4b|\zeta|^2$, $v = -a$ on $|\zeta| = 1$, and $v = 0$ on $|\zeta| = e$. We want to choose a, b so that $\varphi_1 + v \leq \varphi$ on $\widetilde{M} := M \times \{1 \leq |\zeta| \leq e\}$.

On one hand, if $a := ||\varphi_0 - \varphi_1||_{L^\infty(M)}$ then $\varphi_1 + v \leq \varphi$ on $\partial\widetilde{M}$. On the other hand we have (if $b > 0$)

$$(\omega + dd^c(\varphi_1 + v))^{n+1} = (n+1)\omega_{\varphi_1}^n \wedge dd^c v \geq 4b|\zeta|^2\lambda^n(\omega + dd^c|\zeta|^2)^{n+1}.$$

Recall that φ solves

$$(\omega + dd^c\varphi)^{n+1} = 4\varepsilon|\zeta|^2(\omega + dd^c|\zeta|^2)^{n+1}$$

(where $t = \log|\zeta|$). Therefore, if $b := \varepsilon\lambda^{-n}$ we will get $\omega_{\varphi_1+v}^{n+1} \geq \omega_{\varphi}^{n+1}$ and $\varphi_1 + v \leq \varphi$ on $\widetilde{M}$ by comparison principle. It follows that

$$\dot{\varphi}(1) \leq \frac{d}{dt}\left(be^{4t} + (a + b - be^4)t - a - b\right)\Big|_{t=1} = (3e^4 + 1)\frac{\varepsilon}{\lambda^n} + ||\varphi_0 - \varphi_1||_{L^\infty(M)}.$$

Similarly we can show the lower bound for $\dot{\varphi}(0)$ and the estimate follows from the definition of $l(\varphi)$. $\square$

Theorem 1.2 is deduced immediately from Theorem 4.4 and Lemma 4.2, we thus get Theorem 1.1.

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Jagiellonian University, Institute of Mathematics, Łojasiewicza 6, 30-348 Kraków, Poland

E-mail address: `Zbigniew.Blocki@im.uj.edu.pl, umblocki@cyf-kr.edu.pl`