On the L^p Stability for the Complex Monge–Ampère Operator

ZBIGNIEW BŁOCKI*

Introduction

With the standard notation $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$ we have the complex Monge–Ampère operator $(dd^c)^n$ which for smooth functions takes the form

$$(dd^c)^n = n!\, 4^n \det\left(\frac{\partial^2}{\partial z_j \partial \bar{z}_k}\right) d\lambda$$

($d\lambda$ stands for the $2n$-dimensional volume form). In fact, one can well define $(dd^c u)^n$ to be a positive Borel measure if u is plurisubharmonic (PSH, for short) and locally bounded near the boundary of a domain where it is defined (see Demailly [5]).

Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ (i.e., a domain admitting bounded PSH exhaustion function). Take $F \in C(\bar{\Omega})$, $F \geq 0$, and $f \in C(\partial\Omega)$ such that

$$\text{there exists } \chi \in C(\bar{\Omega}) \cap \mathrm{PSH}(\Omega) \text{ such that } \chi|_{\partial\Omega} = f. \tag{1}$$

For example, (1) is always fulfilled if either Ω is strictly pseudoconvex or $f \equiv 0$ (we will be mostly concerned with the latter case). Due to the fundamental result of Bedford and Taylor [2], if Ω is strictly pseudoconvex then there is exactly one solution $u = u_\Omega(f, F)$ to the following Dirichlet problem:

$$\begin{gathered} u \in C(\bar{\Omega}) \cap \mathrm{PSH}(\Omega), \\ (dd^c u)^n = F\, d\lambda, \\ u|_{\partial\Omega} = f. \end{gathered} \tag{2}$$

We extend this result to any hyperconvex Ω (Theorem 1.1). This allows us to state the following.

DEFINITION. Let $1 \leq p, q \leq \infty$. We say that there is *(p,q)-stability* in Ω if there exists a positive constant C, depending only on p, q, and Ω such that for every $F \in C(\bar{\Omega})$ with $F \geq 0$, one has

$$\|u_\Omega(0, F)\|_p \leq C \|F\|_q^{1/n}.$$

Received May 9, 1994.
*Supported by KBN grant 2-1077-91-01.
Michigan Math. J. 42 (1995).

As we show in Proposition 2.1, the notion of (p,q)-stability is independent of Ω. It follows from Hölder's inequality that if there is (p_0, q_0)-stability then there is also (p,q) stability for $p \le p_0$ and $q \ge q_0$. It is also easy to show that there is no $(\infty, 1)$-stability (in the unit ball take $\log|z|$ and consider its regularizations).

Cegrell and Persson, using a connection between real and complex Monge–Ampère operators, have proved $(\infty, 2)$-stability (see [4] and [1]); in [3], $(n,1)$-stability was shown. Here we improve the last result. In order to do it we combine $(\infty, 2)$-stability with a result from [3]. We obtain in particular $(2n, 1)$-stability. Finally we discuss the problem of existence of representation inequalities for PSH functions in B.

This paper was written during my stay in Wuppertal. I am indebted to Professor Klas Diederich for his invitation, to Bergische Universität for their great hospitality, and to the DAAD for financial support.

1. The Dirichlet Problem in Hyperconvex Domains

THEOREM 1.1. *Let Ω be hyperconvex, let $F \in C(\bar{\Omega})$ with $F \ge 0$, and let $f \in C(\partial\Omega)$ fulfill* (1). *Then the function*

$$u := \sup\{v \in \mathrm{PSH} \cap L^\infty(\Omega) : v^*|_{\partial\Omega} \le f,\ (dd^c v)^n \ge F\,d\lambda\}$$

is the unique solution of (2).

See [8] for a more general result in the bidisc.

Proof. First we will show that one may assume that the function χ from (1) has the additional property $(dd^c\chi)^n = 0$ (i.e., χ is maximal). For define

$$\chi' := \sup\{v \in \mathrm{PSH}(\Omega) : v^*|_{\partial\Omega} \le f\}.$$

It follows from the classical potential theory that we can find $h \in C(\bar{\Omega})$, which is harmonic in Ω and equal to f on $\partial\Omega$. This implies that $\chi \le \chi' \le h$, hence $(\chi')^* = (\chi')_* = f$ on $\partial\Omega$. Now it follows from a theorem of Walsh [10] that $\chi' \in C(\bar{\Omega})$ and of course $(dd^c\chi)^n = 0$. We may therefore assume $(dd^c\chi)^n = 0$.

The uniqueness of u follows from the comparison principle [7, Cor. 3.7.4]. Now suppose that F has compact support. By Proposition 1.2 of [6], there exists a smooth, strictly PSH function ψ in Ω such that $\lim_{z \to \partial\Omega} \psi(z) = 0$. Then for some positive A we have

$$(dd^c(\chi + A\psi))^n \ge A^n (dd^c\psi)^n \ge F\,d\lambda,$$

and therefore $\chi + A\psi \le u \le \chi$. This shows that $u^* = u_* = f$ on $\partial\Omega$. Let $\{\Omega_j\}$ be a sequence of strictly pseudoconvex domains such that $\Omega_j \uparrow \Omega$. By [2] we can solve the following problem:

$$u_j \in C(\bar{\Omega}_j) \cap \mathrm{PSH}(\Omega_j),$$
$$(dd^c u_j)^n = F\,d\lambda,$$
$$u_j|_{\partial\Omega_j} = \chi|_{\partial\Omega_j}.$$

By the comparison principle and the fact that $(dd^c\chi)^n = 0$, we have $u \le u_{j+1} \le u_j \le \chi$. We want to show that the sequence $\{u_j\}$ is locally uniformly convergent. Indeed, take $K \subset\subset \Omega$, $\epsilon > 0$, and k_0 such that $K \subset \Omega_{k_0}$ and $|A\psi| \le \epsilon$ on $\partial\Omega_{k_0}$. By the comparison principle for $j, k \ge k_0$ we have

$$\|u_j - u_k\|_K \le \|u_j - u_k\|_{\Omega_{k_0}} \le \|u_j - u_k\|_{\partial\Omega_{k_0}} \le \epsilon,$$

which means that the sequence $\{u_j\}$ is locally uniformly convergent to some $\tilde{u}$. By the convergence theorem [7, Thm. 3.4.3] it is now clear that the $\tilde{u}$ satisfy (2) and so $\tilde{u} = u$.

Now let $F \in C(\bar{\Omega})$, $F \ge 0$, be arbitrary and take $F_j \in C_0(\Omega)$, $F_j \ge 0$, such that $F_j \uparrow F$. If we put $u_j := u_\Omega(f, F_j)$ then, by the comparison principle,

$$|u_j - u_k| \le -u_\Omega(0, |F_j - F_k|) \le 4^{-1} n!^{-1/n} \|F_j - F_k\|_\Omega^{1/n} (R^2 - |z|^2),$$

where R is such that $\Omega \subset B(0, R)$. Therefore

$$\|u_j - u_k\|_\Omega \le \text{const} \|F_j - F_k\|_\Omega^{1/n},$$

which implies that $\{u_j\}$ is uniformly convergent in $\bar{\Omega}$. Theorem 1.1 now follows easily. □

REMARK. It follows from the proof that in Theorem 1.1 we need assume only that $F \in L^2(\Omega)$ instead of $F \in C(\Omega)$.

2. (p, q)-Stability

PROPOSITION 2.1. *(p, q)-stablity in Ω is independent of the domain Ω.*

Proof. It is enough to show that if Ω_1 and Ω_2 are both hyperconvex and $\Omega_1 \subset \Omega_2$ then (p, q)-stability in Ω_2 implies it in Ω_1. Take $F \in C(\bar{\Omega}_1)$, $F \ge 0$, and let $u := u_{\Omega_1}(0, F)$. We can find $F_j \in C(\bar{\Omega}_2)$, $F_j \ge 0$, such that $F_j|_{\Omega_1} = F$ and $F_j \downarrow 0$ in $\bar{\Omega}_2 \backslash \bar{\Omega}_1$. If we put $u_j := u_{\Omega_2}(0, F_j)$ then, by the comparison principle, $u_j \le u \le 0$ in Ω_1 and

$$\|u\|_{p, \Omega_1} \le \|u_j\|_{p, \Omega_2} \le C \|F_j\|_{q, \Omega_2}^{1/n} \to C \|F\|_{q, \Omega_1}^{1/n},$$

which completes the proof. □

The following theorem was proved in [3].

THEOREM 2.2. *Let $u, v \in \mathrm{PSH} \cap L^\infty(\Omega)$ be such that $\lim_{\zeta \to \partial\Omega} u(\zeta) = 0$ and $v \le 0$. Then, for $t \ge 0$,*

$$\int_\Omega |u|^{n+t} (dd^c v)^n \le (t+1) \cdots (t-n) \|v\|_\infty^{n-1} \int_\Omega |v| |u|^t (dd^c u)^n.$$

This theorem was stated in [3], but with the assumption $t = 0$. However, one can easily prove the above version by repeating the arguments from [3].

Substituting $v(z) := |z|^2 - M$ where M is such that $v \le 0$ in Ω, we can easily get $(p, p/n)$-stability for $n \le p \le \infty$. Theorem 2.2 can be also used to prove the following.

THEOREM 2.3. *For $q>1$, (∞,q)-stability implies $(p,pq/(p+nq))$-stability for $nq/(q-1)\le p\le\infty$, and in particular, $(nq/(q-1),1)$-stability.*

Proof. Take $F\in C(\bar\Omega)$, $F\ge 0$, and let $u:=u(0,F)$. Define $v:=u(0,|u|^{p/q})$. Then, by Theorem 2.2, Hölder's inequality, and (∞,q)-stability we have

$$\int_\Omega |u|^p\,d\lambda=\int_\Omega |u|^{p-p/q}(dd^cv)^n\le C_1\|v\|_\infty^n\int_\Omega |u|^{p-p/q-n}F\,d\lambda$$
$$\le C_2\Big(\int_\Omega|u|^p\,d\lambda\Big)^{1/q}\Big(\int_\Omega|u|^p\,d\lambda\Big)^{1-1/q-n/p}\Big(\int_\Omega F^{pq/(p+nq)}\,d\lambda\Big)^{(p+nq)/pq}$$

and the theorem follows. □

Combining the existence of $(\infty,2)$-stability with Theorem 2.3, we arrive at our next result.

COROLLARY 2.4. *There is $(p,2p/(p+2n))$-stability for $2n\le p\le\infty$, in particular $(2n,1)$-stability.*

3. Representation Inequalities in the Unit Ball

Throughout this section we will assume that $n\ge 2$. First we show that, under this assumption, there is no representation formula for PSH functions.

PROPOSITION 3.1. *There is no measurable function $G\colon B\to\mathbb{R}$ such that, for every $u\in \mathrm{PSH}(B)$ with $\lim_{\zeta\to\partial B}u(\zeta)=0$, one has*

$$|u(0)|^n=\int_B G(dd^cu)^n. \tag{3}$$

However, one does have

$$\|u\|_\infty^{n-1}|u(z)|\ge\frac{1}{n!\,(2\pi)^n}\int_B(-\log|T_z|)^n(dd^cu)^n,$$

where $z\in B$ and T_z is a holomorphic automorphism of B such that $T_z(0)=z$ and $T_z^{-1}=T_z$.

Proof. For $z\in B$ let $u:=\log|T_z|$. Then $(dd^cu)^n=(2\pi)^n\delta_z$ (see [7]) and, were (3) fulfilled, we would have $G(z)=(2\pi)^{-n}(-\log|z|)^n$. Substitute next $u(z):=|z|^2-1$ and compute

$$\int_B G(dd^cu)^n=(n-1)!\,n^{-n}<1=|u(0)|^n.$$

The second part of the proposition follows easily from Theorem 2.2. □

We will now focus ourselves on the problem of existence of a positive function G on B such that

$$|u(0)|^n \le \int_B G(dd^c u)^n, \quad u \in \mathrm{PSH}(B), \ \lim_{\zeta \to \partial B} u(\zeta) = 0. \tag{4}$$

(4) implies that

$$|u(z)|^n \le \int_B G \circ T_z (dd^c u)^n, \quad u \in \mathrm{PSH}(B), \ \lim_{\zeta \to \partial B} u(\zeta) = 0.$$

It follows from the proof of Proposition 3.1 that $G(z) \ge (2\pi)^{-n}(-\log|z|)^n$ is a necessary condition for G to fulfill (4). The following example shows, however, that for any positive constant C the function $G(z) := C((-\log|z|)^n + 1)$ cannot fulfill (4).

EXAMPLE 3.2. Take $0 < \epsilon < 1/5$ and for $0 < x < 1$ define $f(x) = (-\log x)^{-1-\epsilon}$. We can find $g \in C^\infty(\mathbb{R})$ such that $0 \le g' \le f'$ in $(0,1)$ and

$$g(x) = \begin{cases} 0 & \text{if } x \le \epsilon/2, \\ f(x) - f(\epsilon) & \text{if } 2\epsilon \le x \le 1/2, \\ f(1/2) - f(\epsilon) & \text{if } 2/3 \le x. \end{cases}$$

Let φ be the function given by $\varphi'(x) = g(x)/x$ and $\varphi(1) = 0$. Since $g' \ge 0$, φ must be logarithmically convex and thus $u(z) := \varphi(|z|^2)$ is PSH. We have

$$\begin{aligned} -u(0) &= \int_0^1 \varphi'(x)\,dx \ge \int_{2\epsilon}^{1/2} (f(x) - f(\epsilon))x^{-1}\,dx \\ &= \epsilon^{-1}((\log 2)^{-\epsilon} - (-\log(2\epsilon))^{-\epsilon}) - (1/2 - 2\epsilon) f(\epsilon). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_B ((-\log|z|^2)^n + 1)(dd^c u)^n \\ &= C_1 \int_0^1 ((-\log x)^n + 1) \frac{d}{dx}(g(x)^n)\,dx \\ &\le C_1 n \int_0^{2/3} ((-\log x)^n + 1) f'(x) f(x)^{n-1}\,dx \le C_2 \int_0^{2/3} (-\log x)^{-1-n\epsilon} x^{-1}\,dx \\ &= C_2 (n\epsilon)^{-1} (\log(3/2))^{-n\epsilon}. \end{aligned}$$

This shows that, for any constant C, the function $G(z) := C((-\log|z|)^n + 1)$ cannot fulfill (4).

A PSH function u is called *radially symmetric* if there exists a C^2 function φ such that $u(z) = \varphi(|z|^2)$. For this special class of PSH functions we have the following.

THEOREM 3.3. *Let $\psi \in C^1((0,1))$ be positive, decreasing, and such that*

$$\lim_{x \to 0} x\psi(x) = \lim_{x \to 1} \psi(x) = 0. \tag{5}$$

Then, for every radially symmetric PSH u in B such that $\lim_{\zeta\to\partial B} u(\zeta)=0$, *we have*

$$|u(0)|^n \le \frac{n}{(4\pi)^n}\left(\int_0^1 \frac{dx}{(-\psi'(x)x^n)^{1/(n-1)}}\right)^{n-1}\int_B \psi(|z|^2)(dd^cu)^n. \tag{6}$$

For instance, the function $\psi(x):=(-\log x)^{n+\epsilon}+(-\log x)^{n-\epsilon}$ fulfills the assumptions of Theorem 3.3 and

$$\int_0^1 \frac{dx}{(-\psi'(x)x^n)^{1/(n-1)}}<\infty.$$

This shows that, for a suitable constant C, the function

$$G(z):=C((-\log|z|)^{n+\epsilon}+(-\log|z|)^{n-\epsilon})$$

fulfills (4) for radially symmetric u. This gives an alternative proof of the result of Persson [9] that for every $q>1$ there is (∞,q)-stability for radially symmetric PSH functions. We do not know whether Theorem 3.3 remains true if we drop the assumption that u is radially symmetric.

Proof of Theorem 3.3. Write $u(z)=\varphi(|z|^2)$. Then

$$\det\left(\frac{\partial^2 u}{\partial z_j}\partial\bar z_k(z)\right)=\varphi'(|z|^2)^{n-1}(\varphi'(|z|^2)+|z|^2\varphi''(|z|^2))$$

and

$$\int_B \psi(|z|^2)(dd^cu)^n=\frac{(4\pi)^n}{n}\int_0^1\psi(x)\,\frac{d}{dx}((x\varphi'(x))^n)\,dx.$$

After integration by parts, by (5) we have

$$\int_B \psi(|z|^2)(dd^cu)^n=-\frac{(4\pi)^n}{n}\int_0^1\psi'(x)x^n\varphi'(x)^n\,dx.$$

Now (6) follows easily from the Hölder inequality. □

References

[1] E. Bedford, *Survey of pluri-potential theory,* Several complex variables, Proceedings of the Mittag-Leffler Institute, 1987–1988 (J. E. Fornaess, ed.), Princeton Univ. Press, Princeton, NJ, 1993.
[2] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation,* Invent. Math. 37 (1976), 1–44.
[3] Z. Błocki, *Estimates for the complex Monge-Ampère operator,* Bull. Polish Acad. Sci. Math. 41 (1993), 151–157.
[4] U. Cegrell and L. Persson, *The Dirichlet problem for the complex Monge-Ampère operator: Stability in L^2,* Michigan Math. J. 39 (1992), 145–151.
[5] J.-P. Demailly, *Potential theory in several complex variables,* preprint, 1991.
[6] N. Kerzman and J.-P. Rosay, *Fonctions plurisousharmoniques d'exhaustion bornées et domaines taut,* Math. Ann. 257 (1981), 171–184.

[7] M. Klimek, *Pluripotential theory,* Oxford Univ. Press, Oxford, 1991.

[8] N. Levenberg and M. Okada, *On the Dirichlet problem for the complex Monge-Ampère operator,* Michigan Math. J. 40 (1993), 507–526.

[9] L. Persson, *On the Dirichlet problem for the complex Monge-Ampère operator,* thesis, Univ. of Umeå, 1992.

[10] J. B. Walsh, *Continuity of envelopes of plurisubharmonic functions,* J. Math. Mech. 18 (1968), 143–148.

Jagiellonian University
Institute of Mathematics
Reymonta 4
30-059 Kraków
Poland