On the $L^p$ Stability for the Complex Monge–Ampère Operator

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Introduction

With the standard notation $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$ we have the complex Monge–Ampère operator $(dd^c)^n$ which for smooth functions takes the form

$$(dd^c)^n = n! \ 4^n \ det \left( \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \right) d\lambda$$

($d\lambda$ stands for the $2n$-dimensional volume form). In fact, one can well define $(dd^c u)^n$ to be a positive Borel measure if $u$ is plurisubharmonic (PSH, for short) and locally bounded near the boundary of a domain where it is defined (see Demailly [5]).

Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$ (i.e., a domain admitting bounded PSH exhaustion function). Take $F \in C(\Omega)$, $F \geq 0$, and $f \in C(\partial \Omega)$ such that

there exists $\chi \in C(\Omega) \cap \text{PSH}(\Omega)$ such that $\chi|_{\partial \Omega} = f$. \hspace{1cm} (1)

For example, (1) is always fulfilled if either $\Omega$ is strictly pseudoconvex or $f \equiv 0$ (we will be mostly concerned with the latter case). Due to the fundamental result of Bedford and Taylor [2], if $\Omega$ is strictly pseudoconvex then there is exactly one solution $u = u_\Omega(f, F)$ to the following Dirichlet problem:

$$u \in C(\Omega) \cap \text{PSH}(\Omega),$$

$$(dd^c u)^n = F \ d\lambda,$$

$$u|_{\partial \Omega} = f.$$ \hspace{1cm} (2)

We extend this result to any hyperconvex $\Omega$ (Theorem 1.1). This allows us to state the following.

DEFINITION. Let $1 \leq p, q \leq \infty$. We say that there is $(p, q)$-stability in $\Omega$ if there exists a positive constant $C$, depending only on $p, q$, and $\Omega$ such that for every $F \in C(\Omega)$ with $F \geq 0$, one has

$$\|u_\Omega(0, F)\|_p \leq C \|F\|_q^{1/n}.$$
As we show in Proposition 2.1, the notion of \((p, q)\)-stability is independent of \(\Omega\). It follows from Hölder's inequality that if there is \((p_0, q_0)\)-stability then there is also \((p, q)\) stability for \(p \leq p_0\) and \(q \geq q_0\). It is also easy to show that there is no \((\infty, 1)\)-stability (in the unit ball take \(\log|z|\) and consider its regularizations).

Cegrell and Persson, using a connection between real and complex Monge-Ampère operators, have proved \((\infty, 2)\)-stability (see [4] and [1]); in [3], \((n, 1)\)-stability was shown. Here we improve the last result. In order to do it we combine \((\infty, 2)\)-stability with a result from [3]. We obtain in particular \((2n, 1)\)-stability. Finally we discuss the problem of existence of representation inequalities for PSH functions in \(B\).

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1. The Dirichlet Problem in Hyperconvex Domains

**Theorem 1.1.** Let \(\Omega\) be hyperconvex, let \(F \in C(\overline{\Omega})\) with \(F \geq 0\), and let \(f \in C(\partial \Omega)\) fulfill (1). Then the function

\[
u := \sup\{v \in PSH \cap L^\infty(\Omega) : v^*|_{\partial \Omega} \leq f, (dd^c v)^n \geq F d\lambda\}
\]

is the unique solution of (2).

See [8] for a more general result in the bidisc.

**Proof.** First we will show that one may assume that the function \(\chi\) from (1) has the additional property \((dd^c \chi)^n = 0\) (i.e., \(\chi\) is maximal). For define

\[
\chi' := \sup\{v \in PSH(\Omega) : v^*|_{\partial \Omega} \leq f\}.
\]

It follows from the classical potential theory that we can find \(h \in C(\overline{\Omega})\), which is harmonic in \(\Omega\) and equal to \(f\) on \(\partial \Omega\). This implies that \(\chi \leq \chi' \leq h\), hence \((\chi')^* = (\chi')_* = f\) on \(\partial \Omega\). Now it follows from a theorem of Walsh [10] that \(\chi' \in C(\overline{\Omega})\) and of course \((dd^c \chi)^n = 0\). We may therefore assume \((dd^c \chi)^n = 0\).

The uniqueness of \(u\) follows from the comparison principle [7, Cor. 3.7.4]. Now suppose that \(F\) has compact support. By Proposition 1.2 of [6], there exists a smooth, strictly PSH function \(\psi\) in \(\Omega\) such that \(\lim_{z \to \partial \Omega} \psi(z) = 0\). Then for some positive \(A\) we have

\[
(dd^c (\chi + A \psi))^n \geq A^n (dd^c \psi)^n \geq F d\lambda,
\]

and therefore \(\chi + A \psi \leq u \leq \chi\). This shows that \(u^* = u_* = f\) on \(\partial \Omega\). Let \(\{\Omega_j\}\) be a sequence of strictly pseudoconvex domains such that \(\Omega_j \uparrow \Omega\). By [2] we can solve the following problem:

\[
u_j \in C(\overline{\Omega}_j) \cap PSH(\Omega_j),
\]

\[
(dd^c u_j)^n = F d\lambda,
\]

\[
u_j|_{\partial \Omega_j} = \chi|_{\partial \Omega_j}.
\]
By the comparison principle and the fact that \((dd^c \chi)^n = 0\), we have \(u \leq u_{j+1} \leq u_j \leq \chi\). We want to show that the sequence \(\{u_j\}\) is locally uniformly convergent. Indeed, take \(K \subset \subset \Omega, \epsilon > 0\), and \(k_0\) such that \(K \subset \Omega_{k_0}\) and \(|A\psi| \leq \epsilon\) on \(\partial \Omega_{k_0}\). By the comparison principle for \(j, k \geq k_0\) we have

\[
\|u_j - u_k\|_K \leq \|u_j - u_k\|_{\Omega_{k_0}} \leq \|u_j - u_k\|_{\partial \Omega_{k_0}} \leq \epsilon,
\]

which means that the sequence \(\{u_j\}\) is locally uniformly convergent to some \(\tilde{u}\). By the convergence theorem [7, Thm. 3.4.3] it is now clear that the \(\tilde{u}\) satisfy (2) and so \(\tilde{u} = u\).

Now let \(F \in C(\bar{\Omega})\), \(F \geq 0\), be arbitary and take \(F_j \in C_0(\Omega), \ F_j \geq 0\), such that \(F_j \uparrow F\). If we put \(u_j := u_{\Omega_j}(f, F_j)\) then, by the comparison principle,

\[
|u_j - u_k| \leq -u_{\Omega}(0, |F_j - F_k|) \leq 4^{-1}n!^{-1/n}\|F_j - F_k\|_{\Omega}^{1/n}(R^2 - |z|^2),
\]

where \(R\) is such that \(\Omega \subset B(0, R)\). Therefore

\[
\|u_j - u_k\|_{\Omega} \leq \text{const}\|F_j - F_k\|_{\Omega}^{1/n},
\]

which implies that \(\{u_j\}\) is uniformly convergent in \(\bar{\Omega}\). Theorem 1.1 now follows easily. 

\[\square\]

Remark. It follows from the proof that in Theorem 1.1 we need assume only that \(F \in L^2(\Omega)\) instead of \(F \in C(\Omega)\).

## 2. \((p, q)\)-Stability

**Proposition 2.1.** \((p, q)\)-stability in \(\Omega\) is independent of the domain \(\Omega\).

**Proof.** It is enough to show that if \(\Omega_1\) and \(\Omega_2\) are both hyperconvex and \(\Omega_1 \subset \subset \Omega_2\) then \((p, q)\)-stability in \(\Omega_2\) implies it in \(\Omega_1\). Take \(F \in C(\bar{\Omega}_1)\), \(F \geq 0\), and let \(u := u_{\Omega_1}(0, F)\). We can find \(F_j \in C(\bar{\Omega}_2)\), \(F_j \geq 0\), such that \(F_j |_{\Omega_1} = F\) and \(F_j \downarrow 0\) in \(\bar{\Omega}_2 \setminus \bar{\Omega}_1\). If we put \(u_j := u_{\Omega_2}(0, F_j)\) then, by the comparison principle, \(u_j \leq u \leq 0\) in \(\Omega_1\) and

\[
\|u\|_{p, \Omega_1} \leq \|u_j\|_{p, \Omega_2} \leq C\|F_j\|_{q, \Omega_2}^{1/n} \rightarrow C\|F\|_{q, \Omega_1}^{1/n},
\]

which completes the proof. \[\square\]

The following theorem was proved in [3].

**Theorem 2.2.** Let \(u, v \in \text{PSH} \cap L^\infty(\Omega)\) be such that \(\lim_{\xi \rightarrow \partial \Omega} u(\xi) = 0\) and \(v \leq 0\). Then, for \(t \geq 0\),

\[
\int_\Omega |u|^{n+t}(dd^c v)^n \leq (t+1) \cdots (t-n)\|v\|_{\infty}^{n-1} \int_\Omega |v||u|'(dd^c u)^n.
\]

This theorem was stated in [3], but with the assumption \(t = 0\). However, one can easily prove the above version by repeating the arguments from [3].

Substituting \(v(z) := |z|^2 - M\) where \(M\) is such that \(v \leq 0\) in \(\Omega\), we can easily get \((p, p/n)\)-stability for \(n \leq p \leq \infty\). Theorem 2.2 can be also used to prove the following.
THEOREM 2.3. For \( q > 1 \), \((\infty, q)\)-stability implies \((p, pq/(p+q))\)-stability for \( nq/(q-1) \leq p \leq \infty \), and in particular, \((nq/(q-1), 1)\)-stability.

Proof. Take \( F \in C(\Omega) \), \( F \geq 0 \), and let \( u := u(0, F) \). Define \( v := u(0, |u|^{p/q}) \). Then, by Theorem 2.2, Hölder’s inequality, and \((\infty, q)\)-stability we have

\[
\int_{\Omega} |u|^p \, d\lambda = \int_{\Omega} |u|^{p-p/q} (dd^c v)^n \leq C_1 \|v\|_{\infty}^n \int_{\Omega} |u|^{p-p/q-n} F \, d\lambda
\]

\[
\leq C_2 \left( \int_{\Omega} |u|^p \, d\lambda \right)^{1/q} \left( \int_{\Omega} |u|^{p/q} \, d\lambda \right)^{1-1/q-p/n} \left( \int_{\Omega} F^{p/(p+q)} \, d\lambda \right)^{(p+q)/pq}
\]

and the theorem follows.

Combining the existence of \((\infty, 2)\)-stability with Theorem 2.3, we arrive at our next result.

COROLLARY 2.4. There is \((p, 2p/(p+2n))\)-stability for \( 2n \leq p \leq \infty \), in particular \((2n, 1)\)-stability.

3. Representation Inequalities in the Unit Ball

Throughout this section we will assume that \( n \geq 2 \). First we show that, under this assumption, there is no representation formula for PSH functions.

PROPOSITION 3.1. There is no measurable function \( G : B \to \mathbb{R} \) such that, for every \( u \in \text{PSH}(B) \) with \( \lim_{z \to \partial B} u(z) = 0 \), one has

\[
|u(0)|^n = \int_B G(dd^c u)^n.
\]

However, one does have

\[
\|u\|_{-1}^n |u(z)| \geq \frac{1}{n! (2\pi)^n} \int_B (-\log|T_z|)^n (dd^c u)^n,
\]

where \( z \in B \) and \( T_z \) is a holomorphic automorphism of \( B \) such that \( T_z(0) = z \) and \( T_z^{-1} = T_z \).

Proof. For \( z \in B \) let \( u := \log|T_z| \). Then \((dd^c u)^n = (2\pi)^n \delta_z \) (see [7]) and, were (3) fulfilled, we would have \( G(z) = (2\pi)^{-n} (-\log|z|)^n \). Substitute next \( u(z) := |z|^2 - 1 \) and compute

\[
\int_B G(dd^c u)^n = (n-1)! \, n^{-n} < 1 = |u(0)|^n.
\]

The second part of the proposition follows easily from Theorem 2.2.

We will now focus ourselves on the problem of existence of a positive function \( G \) on \( B \) such that
\[ |u(0)|^n \leq \int_B G(\ddc u)^n, \quad u \in \text{PSH}(B), \quad \lim_{\xi \to \partial B} u(\xi) = 0. \tag{4} \]

(4) implies that
\[ |u(z)|^n \leq \int_B G \ast T_z(\ddc u)^n, \quad u \in \text{PSH}(B), \quad \lim_{\xi \to \partial B} u(\xi) = 0. \]

It follows from the proof of Proposition 3.1 that \( G(z) \geq (2\pi)^{-n} (-\log |z|)^n \) is a necessary condition for \( G \) to fulfill (4). The following example shows, however, that for any positive constant \( C \) the function \( G(z) := C((-\log |z|)^n + 1) \) cannot fulfill (4).

**Example 3.2.** Take \( 0 < \varepsilon < 1/5 \) and for \( 0 < x < 1 \) define \( f(x) = (-\log x)^{-1-\varepsilon} \). We can find \( g \in C^\infty(\mathbb{R}) \) such that \( 0 \leq g' \leq f' \) in \( (0, 1) \) and
\[ g(x) = \begin{cases} 
0 & \text{if } x \leq \varepsilon/2, \\
(f(x) - f(\varepsilon)) & \text{if } 2\varepsilon \leq x \leq 1/2, \\
(f(1/2) - f(\varepsilon)) & \text{if } 2/3 \leq x. 
\end{cases} \]

Let \( \varphi \) be the function given by \( \varphi'(x) = g(x)/x \) and \( \varphi(1) = 0 \). Since \( g' \geq 0 \), \( \varphi \) must be logarithmically convex and thus \( u(z) := \varphi(|z|^2) \) is PSH. We have
\[ -u(0) = \int_0^1 \varphi'(x) \, dx \geq \int_{2\varepsilon}^{1/2} (f(x) - f(\varepsilon))x^{-1} \, dx \]
\[ = \varepsilon^{-1}((-\log 2)^{-\varepsilon} - (-\log(2\varepsilon))^{-\varepsilon}) - (1/2 - 2\varepsilon) f(\varepsilon). \]

On the other hand,
\[ \int_B ((-\log |z|^2)^n + 1)(\ddc u)^n \]
\[ = C_1 \int_0^1 ((-\log x)^n + 1) \frac{d}{dx} (g(x)^n) \, dx \]
\[ \leq C_1 n \int_0^{2/3} ((-\log x)^n + 1)f'(x) f(x)^{n-1} \, dx \leq C_2 \int_0^{2/3} (-\log x)^{-1-ne} x^{-1} \, dx \]
\[ = C_2 (ne)^{-1}(\log(3/2))^{-ne}. \]

This shows that, for any constant \( C \), the function \( G(z) := C((-\log |z|)^n + 1) \) cannot fulfill (4).

A PSH function \( u \) is called *radially symmetric* if there exists a \( C^2 \) function \( \varphi \) such that \( u(z) = \varphi(|z|^2) \). For this special class of PSH functions we have the following.

**Theorem 3.3.** Let \( \psi \in C^1((0, 1)) \) be positive, decreasing, and such that
\[ \lim_{x \to 0} x \psi(x) = \lim_{x \to 1} \psi(x) = 0. \tag{5} \]
Then, for every radially symmetric PSH $u$ in $B$ such that $\lim_{\xi \to \partial B} u(\xi) = 0$, we have

$$|u(0)|^n \leq \frac{n}{(4\pi)^n} \left( \int_0^1 \frac{dx}{(-\psi'(x)x^n)^{1/(n-1)}} \right)^{n-1} \int_B \psi(|z|^2)(dd^c u)^n.$$  \hspace{1cm} (6)

For instance, the function $\psi(x) := (-\log x)^{n+\epsilon} + (-\log x)^{n-\epsilon}$ fulfills the assumptions of Theorem 3.3 and

$$\int_0^1 \frac{dx}{(-\psi'(x)x^n)^{1/(n-1)}} < \infty.$$

This shows that, for a suitable constant $C$, the function

$$G(z) := C((-\log|z|)^{n+\epsilon} + (-\log|z|)^{n-\epsilon})$$

fulfills (4) for radially symmetric $u$. This gives an alternative proof of the result of Persson [9] that for every $q > 1$ there is $(\infty, q)$-stability for radially symmetric PSH functions. We do not know whether Theorem 3.3 remains true if we drop the assumption that $u$ is radially symmetric.

\textbf{Proof of Theorem 3.3.} Write $u(z) = \varphi(|z|^2)$. Then

$$\det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) \right) = \varphi'(|z|^2)^{-1}(\varphi'(|z|^2) + |z|^2\varphi''(|z|^2))$$

and

$$\int_B \psi(|z|^2)(dd^c u)^n = \frac{(4\pi)^n}{n} \int_0^1 \psi(x) \frac{d}{dx} ((x\varphi'(x))^n) \, dx.$$

After integration by parts, by (5) we have

$$\int_B \psi(|z|^2)(dd^c u)^n = -\frac{(4\pi)^n}{n} \int_0^1 \psi'(x)x^n\varphi'(x)^n \, dx.$$

Now (6) follows easily from the Hölder inequality. \hfill \Box

\section*{References}


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