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Generalizations of the Higher Dimensional Suita Conjecture and Its Relation with a Problem of Wiegerinck

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Abstract
We generalize the inequality being a counterpart of the several complex variables version of the Suita conjecture. For this aim higher order generalizations of the Bergman kernel are introduced. As a corollary some new partial results on the dimension of the Bergman space on pseudoconvex domains are given. A relation between the problem of Wiegerinck on possible dimension of the Bergman space of unbounded pseudoconvex domains in general case and in the case of balanced domains is also shown. Moreover, some classes of domains where the answer to the problem of Wiegerinck is positive are given. Additionally, regularity properties of functions involving the volumes of Azukawa indicatrices are shown.

Keywords Suita conjecture · Bergman kernel · Azukawa indicatrix · Balanced domains · Problem of Wiegerinck

Mathematics Subject Classification 32A36 · 32F45 · 32U35

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1 Introduction

Recall that for the domain $D \subset \mathbb{C}^n$, $w \in D$ we define *the Bergman kernel* K_D (restricted to the diagonal) as follows:

$$K_D(w) := \sup\left\{|f(w)|^2 : f \in \mathcal{O}(D),\ ||f||_D^2 := \int_D |f|^2 \mathrm{d}V \leq 1\right\}. \tag{1}$$

We put $L_h^2(D) := L^2(D) \cap \mathcal{O}(D)$.

Additionally, if $K_D(z) > 0$, then we denote by β_D *the Bergman metric* induced by the Bergman kernel:

$$\beta_D(z; X) := \sqrt{\sum_{j,k=1}^{n} \frac{\partial^2 \log K_D(z)}{\partial z_j \bar{z}_k} X_j \overline{X_k}}, \quad X \in \mathbb{C}^n. \tag{2}$$

We also define *the Azukawa pseudometric* as follows:

$$A_D(w; X) := \exp\left(\limsup_{\lambda \to 0}(G_D(w + \lambda X, w) - \log|\lambda|)\right), \tag{3}$$

$w \in D$, $X \in \mathbb{C}^n$, where $G_D(\cdot, w) = G_w(\cdot)$ denotes *the pluricomplex Green function with the pole at* w.

Denote also *the Azukawa indicatrix at* w:

$$I_D(w) := \{X \in \mathbb{C}^n : A_D(w; X) < 1\}. \tag{4}$$

Recall that a recently obtained version of the higher dimensional version of the Suita conjecture (see [7])

$$K_D(w) \geq \frac{1}{V(I_D(w))}, \quad w \in D, \tag{5}$$

which holds for any pseudoconvex domain may be formulated as follows:

$$K_D(w) \geq K_{I_D(w)}(0), \quad w \in D. \tag{6}$$

Making use of the reasoning as in [4,5,7] we generalize this inequality (see Theorem 2) which then may be applied to get positive results on non-triviality of the Bergman space and its infinite dimensionality (see Sect. 3). Thus it gives a partial solution to a problem of Wiegerinck [20]. He conjectured that the Bergman space of a pseudoconvex domain in $\mathbb{C}^n$ is either zero or infinite dimensional. He showed that the assumption of pseudoconvexity is necessary and that the conjecture is true for $n = 1$ (see also [9]). There are some partial results in higher dimensions: Jucha [14] showed it for some Hartogs domains and Pflug–Zwonek [18] proved it for balanced domains in $\mathbb{C}^2$.

The generalization of the Suita conjecture requires the definition of the higher order Bergman kernels. The introduced objects as well as analogous inequalities have been recently presented in the case of one dimensional domains in the paper [9].

In Sect. 4 we present other classes of domains where the problem of Wiegerinck is solved positively.

In our paper we also present some results that are motivated by the objects that were introduced and studied in the paper [7]; in particular, in Sect. 5 we show regularity properties of the volume of the Azukawa indicatrix.

2 Higher Dimensional Generalization of the Suita Conjecture

Let H be a homogeneous polynomial on $\mathbb{C}^n$ of degree k, $H(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$. We define the operator

$$P_H(f) := \sum_{|\alpha|=k} a_\alpha D^\alpha f, \tag{7}$$

where $f \in \mathcal{O}(D)$ for some domain $D \subset \mathbb{C}^n$.

For the fixed domain $D \subset \mathbb{C}^n$, $z \in D$ we define

$$\begin{aligned} K_D^H(z) := \sup \big\{ &|P_H(f)(z)|^2 : \ f^{(j)}(z) = 0, \\ &j = 0, \ldots, k-1, \quad f \in L_h^2(D), \quad ||f||_D \leq 1 \big\}. \end{aligned} \tag{8}$$

$f^{(j)}(z)$ denotes the jth Frechet derivative of f at z—it is meant here as a homogeneous polynomial of degree j.

Note that

$$K_D^1(z) = K_D(z). \tag{9}$$

For $X \in \mathbb{C}^n$ put $H_X(z) := X_1 z_1 + \cdots + X_n z_n$. If $K_D(z) > 0$, then

$$\beta_D^2(w; X) = \frac{K_D^{H_X}(w)}{K_D^1(w)}. \tag{10}$$

We also put

$$\begin{aligned} K_D^{(k)}(w; X) := K_D^{H_X^k}(w) = \sup \big\{ &|f^{(k)}(w)(X)|^2 : f \in L_h^2(D), \\ &f^{(j)}(w) = 0, \quad j = 0, \ldots, k-1, \quad ||f||_D \leq 1 \big\}. \end{aligned} \tag{11}$$

Note that in the case $n = 1$ we have $K_D^{(k)}(z; 1) = K_D^{(k)}(z)$, where the expression on the right-hand side is understood as in the paper [9].

Following the proof of the analogous result in the case of the Bergman kernel we get the following fundamental properties of K_D^H.

Proposition 1 • *Let $F : D \to G$ be a biholomorphic mapping, and let H be a homogeneous polynomial of degree $k \in \mathbb{N}$, $w \in D$. Then*

$$K_G^H(F(w)) = K_D^{H\circ F'(w)}(w)|\det F'(w)|^2, \tag{12}$$

where $(H \circ F'(w))(X) := H(F'(w)X)$, $X \in \mathbb{C}^n$.

- *Let $D_1, \dots, D_m$ be domains in $\mathbb{C}^n$, $w^j \in D_j$, and let H^j be a homogeneous polynomial on $\mathbb{C}^n$. Then*

$$K_{D_1\times\ldots\times D_m}^{H_1\times\ldots\times H_m}(w^1, \dots, w^m) = K_{D_1}^{H_1}(w^1)\cdot\dots\cdot K_{D_m}^{H_m}(w^m). \tag{13}$$

- *If D is a balanced pseudoconvex domain, H is a homogeneous polynomial on $\mathbb{C}^n$, then*

$$K_D^H(0) = \frac{|P_H(H^*)|^2}{||H||_D^2} = \frac{\left|\sum_{|\alpha|=k} |a_\alpha|^2\alpha!\right|^2}{\int_D |H(z)|^2 dV(z)}, \tag{14}$$

where $H^*(z) = \sum_{|\alpha|=k} \bar{a}_\alpha z^\alpha$.

To make the presentation simpler we shall often assume that the point (pole of the Green function) will be $w = 0$. In such a case we denote $D_a := e^{-a}\{G < a\}$ for $a \le 0$. Additionally, put $D_{-\infty} := I_D(0)$. We shall often use the obvious fact that the sets $\{G < a\}$ and D_a are linearly isomorphic, $-\infty < a \le 0$.

The properties of the Green function give the equality $(D_a)_b = D_{a+b}$ for $-\infty \le a, b \le 0$. Note also that $K_{D_a}^H(0) = e^{2(n+k)a}K_{\{G<a\}}^H(0)$.

Our main result is the following.

Theorem 2 *Let D be a pseudoconvex domain in $\mathbb{C}^n$, $w = 0 \in D$ and let H be a homogeneous polynomial of degree k. Then the function*

$$[-\infty, 0] \ni a \to K_{D_a}^H(0) \tag{15}$$

is non-decreasing.

In particular, $K_{I_D(0)}^H(0) \le K_D^H(0)$. Consequently, $K_{I_D(0)}^{(k)}(0; X) \le K_D^{(k)}(0; X)$ for any $X \in \mathbb{C}^n$.

Proof We compile the reasoning as in the proof of Theorem 1 in [4], the proof of Theorem 6.3 in [5] and the proof of Theorem 1 in [7].

If D_j is a sequence of domains in $\mathbb{C}^n$ such that $D_j \subset D_{j+1}$ and $D = \bigcup D_j$, then $K_{D_j}^H$ and G_{D_j} decrease to K_D^H and G_D, respectively. Without loss of generality we may thus assume that D is a bounded hyperconvex domain.

The properties of the Green function and thus the ones of the sets D_a reduce the problem of the monotonicity of (15) to the proof of the inequality $K_D^H(0) \ge K_{D_a}^H(0)$ for a fixed $a < 0$.

The main tool in the proof will be the following L^2-estimate for $\bar\partial$ due to Donelly and Fefferman (see [11] or Theorem 2.2 in [5]): if α is a (0, 1)-form in a pseudoconvex domain D with coefficients in $L^2_{\text{loc}}(D)$ such that $\bar\partial\alpha = 0$, φ is plurisubharmonic in

D and ψ is of the form $\psi = -\log(-v)$, where v is negative plurisubharmonic in D, then there exists $u \in L^2_{\mathrm{loc}}(D)$ solving $\bar{\partial} u = \alpha$ and satisfying the estimate

$$\int_D |u|^2 e^{-\varphi} \mathrm{d}\lambda \leq C \int_D h e^{-\varphi} \mathrm{d}\lambda, \tag{16}$$

where $C > 0$ is an absolute constant (in fact the optimal one is $C = 4$) and $h \geq 0$ is such that $i\overline{\alpha} \wedge \alpha \leq h\, i\partial\bar{\partial}\psi$.

Take any $f \in L^2_h(\{G < a\})$ with $f^{(j)}(0) = 0$, $j = 0, \ldots, k-1$. We will use the Donnelly–Fefferman estimate with the following data:

$$\varphi := 2(n+k+1)G,\ \ \psi := -\log(-G),\ \ \alpha := \bar{\partial}(f\chi \circ G), \tag{17}$$

where

$$\chi(t) := \begin{cases} 0, & t \geq a, \\ \int_{-a}^{-t} \frac{e^{-(n+k+1)s}}{s} \mathrm{d}s, & t < a. \end{cases} \tag{18}$$

Since

$$i\overline{\alpha} \wedge \alpha \leq |f|^2 (\chi' \circ G)^2 G^2 i\partial\bar{\partial}\psi,$$

by (16) we can find u with $\bar{\partial} u = \alpha$ and

$$\int_D |u|^2 e^{-2(n+k+1)G} \mathrm{d}\lambda \leq C \int_D |f|^2 (\chi' \circ G)^2 G^2 e^{-2(n+k+1)G} \mathrm{d}\lambda. \tag{19}$$

Then the holomorphic function

$$F := f\chi \circ G - u \tag{20}$$

satisfies $F^{(j)}(0) = 0$ (since near the origin $e^{-2(n+k+1)G} \geq \delta|z|^{-2(n+k+1)}$ for some $\delta > 0$) and $P_{D,H}(F)(0) = \chi(-\infty)P_{D,H}(f)(0) = Ei(-(n+k+1)a)P_{D,H}(f)(0)$. Moreover,

$$||F||_{L^2(D)} \leq (\chi(-\infty) + \sqrt{C})||f||_{\{G<a\}}, \tag{21}$$

which implies that

$$K_D^H(0) \geq c(n,a,k) K_{\{G<a\}}^H(0), \tag{22}$$

where $c(n,a,k) = \frac{Ei(-(n+k+1)a)^2}{(Ei(-(n+k+1)a)+\sqrt{C})^2}$.

We can now use the tensor power trick: for large m we consider the domain $D^m \subset \mathbb{C}^{nm}$. Using the formulas for the Bergman kernel and the Green function for product domains and letting m tend to ∞ we will get

$$K_D^H(0) \geq e^{2(n+k)a} K_{\{G<a\}}^H(0) = K_{D_a}^H(0). \tag{23}$$

Similarly, as in [7] we note that the continuity of the Azukawa metric (and the existence of the limit in its definition)—see [21,22]—implies the convergence in the sense of Hausdorff: $D_a \to I_D(w)$ which together with basic properties of the Bergman functions implies the desired inequality. □

Remark 3 It would be interesting to verify whether the function

$$(-\infty, 0] \ni a \to \log K^H_{D_a}(w) \tag{24}$$

is convex as it is in the case of $H \equiv 1$ (see final remark in [6]).

Note that the non-triviality of the space $L^2_h(D)$ is equivalent to the fact that for any $w \in D$ there are a k and X such that $K^{(k)}_D(w; X) > 0$.

The infinite dimensionality of $L^2_h(D)$ is equivalent to the existence for any (equivalently, some) $w \in D$ a subsequence (k_ν) and a sequence (X^ν) such that $K^{(k_\nu)}_D(w; X^\nu) > 0$. Therefore, we conclude

Proposition 4 *Let D be a pseudoconvex domain in $\mathbb{C}^n$.*

- *If for some $w \in D$ the space $L^2_h(I_D(w))$ is not trivial, then so is the space $L^2_h(D)$.*
- *If for some $w \in D$ the dimension of $L^2_h(I_D(w))$ is infinite, then so is the dimension of $L^2_h(D)$.*

In fact, one may also conclude from Theorem 2 a more precise version of Proposition 4.

Corollary 5 *Let D be a pseudoconvex domain in $\mathbb{C}^n$, $w \in D$, $-\infty < a \le 0$. Then*

$$\dim(L^2_h(I_D(w))) \le \dim(L^2_h(D_a(w))). \tag{25}$$

Making use of the result from [18] we get the following partial solution of the problem of Wiegerinck (see [20]).

Corollary 6 *Let D be a pseudoconvex domain in $\mathbb{C}^2$. If for some $w \in D$ the space $L^2_h(I_D(w))$ is not trivial, then the dimension of $L^2_h(D)$ is infinite.*

Note that the non-triviality of the space $L^2_h(I_D(w))$ in the case $n = 2$ is precisely described in [18].

Remark 7 To answer the problem of Wiegerinck in dimension two it would be then sufficient to decide what the dimensions of $L^2_h(D)$ are in the case when $L^2_h(I_D(w)) = \{0\}$ for all $w \in D$. The solution of that problem seems to be very probable to get. Perhaps one should start with the solution of the problem when $A_D \equiv 0$, or $G \equiv -\infty$?

3 On the Finite Dimensional Bergman Space on D_a

Note that Corollary 5 leaves the problem on the mutual relation between the dimensions of the spaces $L^2_h(D_a)$ for different a open. Note that the restriction: $L^2_h(D_b) \ni f \to f(e^{a-b}\cdot)_{|D_a} \in L^2_h(D_a)$, $-\infty < a < b \le 0$ gives the inequality

$$\dim(L^2_h(D_a)) \le \dim(L^2_h(D_b)). \tag{26}$$

In fact, we shall prove that the equality holds.

Proposition 8 *Let D be a pseudoconvex domain in $\mathbb{C}^n$, $0 \in D$. Then for any $-\infty < a \leq 0$ the dimension of $L_h^2(D_a)$ is the same.*

Proof It is sufficient to show that if $-\infty < a < 0$, then the dimension of $L_h^2(D_a)$ is equal to that of $L_h^2(D)$. To prove this it is sufficient to show that if we get the system $\{f_1, \dots, f_N\}$ of linearly independent elements of $L_h^2(D_a)$, then there are elements $F_1, \dots, F_N$ from $L_h^2(D)$ linearly independent. For the functions f_l we follow a construction from the proof of Theorem 2. First we choose k so big that the functions $\tilde{f}_l$, $l = 1, \dots, N$, are linearly independent in the space of polynomials, where $\tilde{f}_l(z) := \sum_{m=0}^k \frac{f_l^{(m)}(0)}{m!}(z)$, $l = 1, \dots, N$. Fix now a smooth function $\chi : [-\infty, 0] \to [0, 1]$ such that χ equals 1 near $-\infty$ and $\chi(t) = 0$, $t \geq a$. Now starting with the functions f_l we proceed with the construction of functions F_l as in the proof of Theorem 2 with $\varphi := 2(n + k + 1)G$ and the mapping χ. The functions F_l are L_h^2 functions on D that satisfy the equality

$\tilde{f}_l \equiv \tilde{F}_l$, $l = 1, \dots, N$, which implies immediately the linear independence of F_l, $l = 1, \dots, N$. □

Remark 9 Proposition 8 together with Corollary 5 suggests that the equality of dimensions of all Bergman spaces $L_h^2(D_a)$, $-\infty \leq a \leq 0$ may hold, which in turn would reduce the problem of Wiegerinck from the general case to that in the class of pseudoconvex balanced domains (the set $D_{-\infty}$).

Remark 10 Note that the results presented in this section imply that if $L_h^2(D)$ is finite dimensional, then all the functions lying in $L_h^2(\{G < a\})$, $-\infty < a < 0$ are the restrictions of the functions from $L_h^2(D)$—this very special phenomenon is a fact which may serve as another hint that the problem of Wiegerinck should have a positive answer.

4 Other Sufficient Conditions for the Positive Solution of the Problem of Wiegerinck

In this section we shall present two other sufficient conditions on domains that guarantee that the domain from the given class will give the positive answer to the problem of Wiegerinck. It should be noted however that it is probably not so easy to check whether assumptions of the next two results are satisfied in specific cases of unbounded domains.

Consequently, we have no examples of domains that would satisfy the assumptions of next two theorems for which the solution of the problem of Wiegerinck could not be concluded from other known criteria. Therefore, it would be interesting if one could find such examples.

Theorem 11 *Let D be a pseudoconvex domain in $\mathbb{C}^n$ such that for some $w \in D$ and $a \leq 0$ the sublevel set $\{G_D(\cdot, w) < a\}$ does not satisfy the Liouville property, that* *is there exists a bounded non-constant holomorphic function defined there. Then the Bergman space $L^2_h(D)$ is either trivial or infinite dimensional.*

Proof Assume that there exists non-zero $f \in L^2_h(D)$. There exists $k \geq 0$ such that $f^{(j)}(w) = 0$ for $j = 0, 1, \dots, k-1$ but $f^{(k)}(w) \neq 0$. We can also find Q holomorphic and bounded in $\{G < a\}$, where $G = G_D(\cdot, w)$, and $m \geq 1$ such that $Q^{(j)}(w) = 0$ for $j = 0, 1, \dots, m-1$ but $Q^{(m)}(w) \neq 0$. For $l \geq 1$ define

$$\alpha := \bar\partial(Q^l f \chi \circ G) = Q^l f \chi' \circ G \,\bar\partial G,$$

where $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ is such that $\chi(t) = 1$ for $t \leq b$ and $\chi(t) = 0$ for $t \geq c$, where b and c are such that $b < a < c < 0$. Set

$$\varphi := 2(n + k + lm)G, \quad \psi := -\log(-G),$$

then

$$i\bar\alpha \wedge \alpha \leq |Q|^{2l}|f|^2(\chi' \circ G)^2 i\partial G \circ \bar\partial G \leq |Q|^{2l}|f|^2(\chi' \circ G)^2 G^2 i\partial\bar\partial\psi$$

and by the Donnelly–Fefferman estimate there exists $u \in L^2_{\text{loc}}(D)$ with $\bar\partial u = \alpha$ and

$$\begin{aligned} ||u||^2 &\leq \int_D |u|^2 e^{-\varphi} \mathrm{d}\lambda \\ &\leq 4 \int_D |Q|^{2l}|f|^2(\chi' \circ G)^2 G^2 e^{-2(n+k+lm)G} \mathrm{d}\lambda \leq C||f||^2. \end{aligned} \tag{27}$$

Set $F = Q^l f \chi \circ G - u$. Then $F \in L^2_h(D)$ and $F^{(j)}(w) = 0$ for $j = 0, \dots, k+lm-1$, but $F^{(k+lm)}(w) \neq 0$. Since l is arbitrary, it follows that $L^2_h(D)$ is infinite dimensional. □

Theorem 12 *Let D be a pseudoconvex domain in $\mathbb{C}^n$ and $w_j \in D$ an infinite sequence, not contained in any analytic subset of D, and such that for every $j \neq k$ there exists $t < 0$ such that $\{G_j < t\} \cap \{G_k < t\} = \emptyset$, where $G_j := G_D(\cdot, w_j)$. Then $L^2_h(D)$ is either trivial or infinite dimensional.*

Proof Assume that $f \in L^2_h(D)$, $f \not\equiv 0$. Choosing a subsequence if necessary we may assume that $f(w_j) \neq 0$ for all j. For every k we want to construct $F \in L^2_h(D)$ such that $F(w_j) = 0$ for $j = 1, \dots, k-1$ but $F(w_k) \neq 0$. It will then follow that $L^2_h(D)$ is infinite dimensional.

We can find $t_k < 0$ such that $\{G_j < t_k\} \cap \{G_l < t_k\} = \emptyset$ for $j, l = 1, \dots, k$, $j \neq l$. Set $G := G_1 + \dots + G_{k-1}$ and

$$\alpha := \bar\partial\big(f \chi \circ G\big) = f \chi' \circ G \,\bar\partial G,$$

where $\chi \in C^\infty(\mathbb{R})$ is such that $\chi(t) = 0$ for $t \leq (k-1)t_k - 2$ and $\chi(t) = 0$ for $t \geq (k-1)t_k - 1$. Define the weights

$$\varphi := 2n(G + G_k), \quad \psi := -\log(-G),$$

we then have

$$i\bar{\alpha} \wedge \alpha \leq |f|^2 G^2 (\chi' \circ G)^2 i\partial\bar{\partial}\psi.$$

By the Donnelly–Fefferman estimate we can find $u \in L^2_{\text{loc}}(D)$ with $\bar{\partial}u = \alpha$, satisfying the estimate

$$||u||^2 \leq \int_D |u|^2 e^{-\varphi} d\lambda \leq 4 \int_D |f|^2 G^2 (\chi' \circ G)^2 e^{-\varphi} d\lambda.$$

For every $z \in D$ with $G(z) < (k-1)t_k$ there exists $j \leq k-1$ such that $G_j(z) < t_k$, and therefore $G_k \geq t_k$ on $\{G \leq (k-1)t_k - 1\}$. It follows that $||u|| < \infty$ and thus $F := f\chi \circ G - u \in L^2_h(D)$. Since $e^{-\varphi}$ is not locally integrable near $w_1, \ldots, w_k$, we conclude that $F(w_1) = \cdots = F(w_{k-1}) = 0$ and $F(w_k) = f(w_k)$ (the latter since $G(w_k) \geq (k-1)t_k$). $\square$

5 Regularity of the Volume of the Azukawa Indicatrix

For $k \geq 1$ we define the kth *order Carathéodory–Reiffen pseudometric* as follows:

$$\gamma_D^{(k)}(z; X) := \sup \left\{ \left| f^{(k)}(z)X/k! \right|^{1/k} : f^{(j)}(z) = 0,\, j = 0, \ldots, k-1 \right\}, \tag{28}$$

$z \in D,\ X \in \mathbb{C}^n$ and the supremum is taken over all holomorphic $f : D \to \mathbb{D}$ where $\mathbb{D}$ is the unit disc in $\mathbb{C}$.

Recall that the bounded domain $D \subset \mathbb{C}^n$ is called *strictly hyperconvex* if there are a bounded domain $\Omega \subset \mathbb{C}^n$, a continuous plurisubharmonic function $u : \Omega \to (-\infty, 1)$ such that $D = \{u < 0\}$, u is exhaustive for Ω and for all $c \in [0, 1]$ the set $\{u < c\}$ is connected (see [17]). It is elementary to see that $\gamma_D^{(k)} \leq A_D$. In general, the function A_D is upper semicontinuous (see [12]) and in the case of the hyperconvex D even continuous (see [21]).

It follows directly from the definition that the functions $D \times \mathbb{C}^n \ni (z; X) \to \gamma_D^{(k)}(z; X)$ are logarithmically plurisubharmonic. Recall that for a strictly hyperconvex D and for any $z \in D$ we have the convergence $\lim_{k\to\infty} \gamma_D^{(k)}(z; X) = A_D(z; X)$ for almost all $X \in \mathbb{C}^n$ (Theorem 1 in [17]). Consequently, for strictly hyperconvex domain D we get that the function $D \times \mathbb{C}^n \ni (z; X) \to A_D(z; X)$ is logarithmically plurisubharmonic. Since any pseudoconvex domain D can be exhausted by an increasing sequence of strictly hyperconvex domains $(D_\nu)_\nu$, the Azukawa pseudometric A_D is the decreasing limit $\lim_{\nu\to\infty} A_{D_\nu}$ we deduce the following

Proposition 13 *Let D be a pseudoconvex domain in $\mathbb{C}^n$. Then* $\log A_D$ *is plurisubharmonic* (*as a function defined on $D \times \mathbb{C}^n$*).

For the pseudoconvex domain $D \subset \mathbb{C}^n$ define the following pseudoconvex (see e. g. [13] and use the logarithmic plurisubharmonicity of A_D) Hartogs domain with the basis D and balanced fibers

$$\Omega_D := \{(z; X) \in D \times \mathbb{C}^n : A_D(z; X) < 1\}. \tag{29}$$

Consequently, making use of Theorem 1.4 from [3] ($\Omega_D(z) = I_D(z)$) we get the following result.

Theorem 14 *Let D be a pseudoconvex domain in $\mathbb{C}^n$ then the function*

$$D \ni z \to -\log V(I_D(z)) \tag{30}$$

is plurisubharmonic.

It is natural to ask the question on the logarithmic convexity of A_D in the case when D is convex. It turns out that the answer is positive.

Theorem 15 *Let D be a convex domain in $\mathbb{C}^n$. Then the function*

$$D \ni z \to -\log V(I_D(z)) \tag{31}$$

is convex.

Proof Due to the Lempert theorem (see e. g. [16])) we have the equality $A_D = \kappa_D$, where κ_D is the Kobayashi pseudometric of D. Without loss of generality we may assume that D is bounded. Let $t \in [0, 1]$, $w, z \in D$. We claim that $tI_D(w) + (1 - t)I_D(z) \subset I_D(tw + (1 - t)z)$. Actually, let $X \in I_D(w)$, $Y \in I_D(z)$. Then there are analytic discs $f, g : \mathbb{D} \to D$ such that $f(0) = w$, $g(0) = z$, $f'(0) = X$, $g'(0) = Y$. Consequently, the mapping $h := tf + (1-t)g$ maps $\mathbb{D}$ into D, $h(0) = tw + (1-t)z$, $h'(0) = tX + (1 - t)Y$, so $tX + (1 - t)Y \in I_D(tw + (1 - t)z)$.

It follows from the Brunn–Minkowski inequality that the Lebesgue measure is logarithmically concave (see e. g. [19]); therefore,

$$\begin{aligned} V(I_D(tw + (1 - t)z) &\geq V(tI_D(w) + (1 - t)I_D(z)) \\ &\geq V(I_D(w))^t V(I_D(z))^{1-t} \end{aligned} \tag{32}$$

which finishes the proof. □

The higher dimensional Suita conjecture (i. e. the inequality (5)) may also be presented in the following way:

$$F_D(w) := \sqrt[n]{K_D(w) \cdot V(I_D(w))} \geq 1, \quad w \in D. \tag{33}$$

Note that the function F_D has the following properties:

- F is biholomorphically invariant,
- if D is a bounded pseudoconvex balanced domain, then $F_D(0) = 1$.

The explicit formulas for the function F_D (see [7,8]) may be used to study the boundary behavior of F_D. Recently, the case of the strongly pseudoconvex domains was completely solved.

Proposition 16 (see [1]) *Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$. Then* $\lim_{z\to\partial D} F_D(z) = 1$.

Note that the above property follows directly from a recent result from [10] (see Theorem 4.1 in [15]).

The recent paper [2] is devoted to the study of the boundary behavior of the (formally slightly modified) function F_D on a wider class of domains.

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