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On the uniform estimate in the Calabi-Yau theorem, II

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Abstract We show that a pluripotential proof of the uniform estimate in the Calabi-Yau theorem works also in the Hermitian case.

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Tosatti and Weinkove [15] recently proved a general L^∞-estimate for the complex Monge-Ampère equation on compact Hermitian manifolds. This gave, using estimates proved earlier in [6, 9, 8, 14], a generalization of the Calabi-Yau theorem [16] to the Hermitian case. Subsequently, the estimate from [15] was improved (with a different proof) by Dinew and Kołodziej [7]. The aim of this note is to give yet another proof of this estimate. We will show that in fact a very simple modification of the proof for Kähler manifolds from [4] gives the required result.

We assume that M is a compact complex manifold of complex dimension n equipped with Hermitian form ω. We will give a simple proof of the following estimate shown already in [7] (where the method from [11] was used):

Main Theorem. *Assume that $\varphi \in C^2(M)$ is such that $\omega + dd^c\varphi \geqslant 0$ and*

$$(\omega + dd^c\varphi)^n = f\omega^n.$$

Then for $p > 1$,

$$\operatorname{osc}\varphi \leqslant C(M, \omega, p, \|f\|_{L^p(M)}).$$

Our proof, as in [4], will use the local L^q-stability for the complex Monge-Ampère operator which is quite easy for $q = 2$ (it is due to Cheng and Yau) and much more involved for $q > 1$ (proved by Kołodziej [10]). We will thus obtain a very simple proof of the above result for $p > 2$, and for arbitrary $p > 1$, we will have to use Kołodziej's local estimate (it is hidden in Proposition 2 below, see also Remark 2 below).

In the proof we will use the following two local results:

Proposition 1 [3]. *Let Ω be a bounded domain in $\mathbb{C}^n$. Suppose that u, v are continuous functions on $\bar{\Omega}$ such that $u \leqslant v$ on $\partial\Omega$, u is plurisubharmonic in Ω, $v \in C^2(\Omega)$. Assume moreover that on the set $\{dd^c v > 0\}$ we have $(dd^c v)^n \leqslant (dd^c u)^n$. Then $u \leqslant v$ in Ω.*

Proposition 2 [4]. *Let u be a negative C^2 plurisubharmonic function in a bounded domain Ω in $\mathbb{C}^n$. Let $a > 0$ be such that the sublevel set $\{u < \inf_\Omega u + a\}$ is relatively compact in Ω. Then for $p > 1$, we have*

$$\|u\|_{L^\infty(\Omega)} \leqslant C(n, \operatorname{diam}(\Omega), a, \|u\|_{L^1(\Omega)}, p, \|f\|_{L^p(\Omega)}),$$

where $f = \det(u_{j\bar{k}})$.

Remark 1. The generalized comparison principle Proposition 1 for the real Monge-Ampère was proved in [12], the proof in [3] is essentially a repetition of that argument. In the complex case, in a slightly weaker form than here, it was first proved in [13]. The inequality

$$\partial^2(u + \varepsilon\psi - v)(z_0 + \zeta\alpha)/\partial\zeta\partial\bar{\zeta}(0) > 0$$

at the end of proof of Theorem 3.7 in [3] should be understood in the weak sense, namely that the function $\zeta \mapsto (u + \varepsilon\psi - v)(z_0 + \zeta\alpha)$ is strongly subharmonic near 0.

Remark 2. The main tool in the proof of Proposition 2 in [4] is the following counterpart of the Aleksandrov-Bakelman estimate from the real case: if v is a C^2 plurisubharmonic function in Ω vanishing on $\partial\Omega$, then for $q > 1$, we have

$$\|v\|_{L^\infty(\Omega)} \leqslant C(n, q, \operatorname{diam}(\Omega))\|f\|^{1/n}_{L^q(\Omega)}, \tag{1}$$

where $f = \det(v_{j\bar{k}})$. For $q = 2$, it was proved by Cheng and Yau (see [1,5], and also [4]) and for arbitrary $q > 1$ by Kołodziej [10]. The inequality (1) is not stated explicitly in [10] but it can be easily deduced from the proof of Theorem 3 in [10]. To see that the constant depends only on the diameter of Ω, let B be a ball containing it and consider $\widetilde{v}$ plurisubharmonic and continuous in B, vanishing on ∂B and such that $\det(\widetilde{v}_{j\bar{k}}) = \widetilde{f}$ (in the weak sense of [2]), where

$$\widetilde{f} = \begin{cases} f, & \text{in } \Omega, \\ 0, & \text{in } B\backslash\Omega. \end{cases}$$

Then $\widetilde{v} \leqslant v \leqslant 0$ in Ω by the comparison principle and we get

$$\|v\|_{L^\infty(\Omega)} \leqslant \|\widetilde{v}\|_{L^\infty(B)} \leqslant C\|\widetilde{f}\|^{1/n}_{L^q(B)} = C\|f\|^{1/n}_{L^q(\Omega)}.$$

Proof of Main Theorem. Assume that $\max_M \varphi = 0$. Choose $y \in M$, where φ attains minimum. We can find a local potential g near y such that

$$\frac{1}{C}dd^c g \leqslant \omega \leqslant C dd^c g \tag{2}$$

for some uniform constant $C > 0$ (depending only on M and ω). The estimate

$$\|\varphi\|_{L^1(M)} \leqslant C(M, \omega) \tag{3}$$

follows easily from local properties of the plurisubharmonic function $Cg + \varphi$ and a finite number of other similar plurisubharmonic functions on a finite number of charts covering M (for this it is enough to use only that they are subharmonic).

Similarly as in [4], using the Taylor expansion of g about y, we can find $a, r > 0$, depending only on M and ω, and $g \in C^\infty(\overline{B}(y,r))$ satisfying (2) and such that $g < 0$, g attains minimum at y, and $g \geqslant g(y) + 2a$ on $\partial B(y,r)$ (where $B(y,r)$ is a ball centered at y with radius r in local Euclidean coordinates).

By [2] there exists $u \in C(\overline{B}(y,r))$, plurisubharmonic in $B(y,r)$, such that $u = (1/C)g + \varphi$ on $\partial B(y,r)$ and $(dd^c u)^n = f\omega^n$ in $B(y,r)$. We have

$$Cg + \varphi \leqslant u \leqslant \frac{1}{C}g + \varphi, \tag{4}$$

where the first inequality follows from the standard comparison principle [2] and the second one from Proposition 1. From (3) and the first inequality in (4), it follows that $\|u\|_{L^1(B(y,r))}$ is under control. By the second inequality in (4) on $\partial B(y,r)$, we have

$$u = \varphi + \frac{1}{C}g \geqslant \varphi(y) + \frac{1}{C}(g(y) + 2a) \geqslant \inf_{B(y,r)} u + \frac{2a}{C},$$

and therefore, $\{u < \inf_\Omega u + a/C\}$ is relatively compact in $B(y,r)$. Proposition 2 now implies that $\|u\|_{L^\infty(B(y,r))}$ is under control, and the required estimate follows. □

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