Ohsawa-Takegoshi Extension Theorem and Applications

Zbigniew Błocki

(Jagiellonian University, Kraków, Poland)

http://gamma.im.uj.edu.pl/~blocki

Institute of Mathematics, Chinese Academy of Sciences

Beijing, 31 October 2008

Theorem (Ohsawa-Takegoshi, 1987).
$\Omega \subset \mathbb{C}^n$ bounded, pseudoconvex, $\varphi \in PSH(\Omega)$,
H hyperplane in $\mathbb{C}^n$, $\Omega' := \Omega \cap H$,
$f \in \mathcal{O}(\Omega')$

$\Rightarrow \exists\ F \in \mathcal{O}(\Omega)$ s.th. $F|_{\Omega'} = f$ and

$$\int_{\Omega} |F|^2 e^{-\varphi} d\lambda \le C(n, \operatorname{diam}\Omega) \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

Original proof: L^2-theory of $\overline{\partial}$-equation on complete Kähler manifolds + commutator identities (in the spirit of Bochner, Kodaira, Nakano...).

Berndtsson (1996): proof without employing any Kähler metrics; if $H = \{z_1 = 0\}$ and $\Omega \subset \{|z_1| < 1\}$ then $C = 4\pi$.

Bergman kernel

$$K_\Omega := \sup\{|f|^2 : f \in \mathcal{O}(\Omega), \int_\Omega |f|^2 d\lambda \le 1\}$$

(on the diagonal)

O-T implies that

$$K_{\Omega'} \le C K_\Omega \;\text{ on } \Omega'.$$

Corollary (original motivation behind O-T). Ω bounded, psedoconvex, with C^2 boundary. Then

$$K_\Omega(z) \ge \frac{1}{C(\text{dist}\,(z, \partial\Omega))^2}, \quad z \in \Omega,$$

for some $C = C(\Omega) > 0$.

$\varphi \in PSH(\Omega)$, i.e. $\left(\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}\right) \geq 0$

Lelong number of φ at z_0:

$$\nu_\varphi(z_0) = \varliminf_{z \to z_0} \frac{\varphi(z)}{\log|z - z_0|} = \lim_{r \to 0^+} \frac{\varphi^r(z_0)}{\log r},$$

where for $r > 0$

$$\varphi^r(z) := \max_{\overline{B}(z,r)} \varphi$$

$(z \in \Omega_r := \{z \in \Omega : \mathrm{dist}\,(z, \partial\Omega) > r\})$.

One can show that φ^r is a plurisubharmonic continuous function in Ω_r, decreasing to φ as r decreases to 0.

Demailly Approximation

Theorem (Demailly, 1992). $\varphi \in PSH(\Omega)$,

$$\varphi_m := \frac{1}{2m} \log K_{\Omega, e^{-2m\varphi}}, \quad m = 1, 2 \ldots,$$

where

$$K_{\Omega, e^{-2m\varphi}} = \sup\{|f|^2 : f \in \mathcal{O}(\Omega),\ \int_\Omega |f|^2 e^{-2m\varphi} \leq 1\}.$$

$\Rightarrow \quad \exists\, C_1, C_2 > 0$ depending only on Ω s.th.

$$\varphi - \frac{C_1}{m} \leq \varphi_m \leq \varphi^r + \frac{1}{m} \log \frac{C_2}{r^n} \quad \text{in} \;\; \Omega_r.$$

In particular, $\varphi_m \to \varphi$ pointwise and in $L^1_{loc}(\Omega)$.
Moreover,

$$\nu_\varphi - \frac{n}{m} \leq \nu_{\varphi_m} \leq \nu_\varphi \quad \text{in} \;\; \Omega.$$

Proof: For a fixed $z \in \Omega$, by O-T (extension from a single point to Ω) one can find $f \in \mathcal{O}(\Omega)$ s.th.

$$\int_\Omega |f|^2 e^{-2m\varphi} d\lambda \le C|f(z)|^2 e^{-2m\varphi(z)} = 1.$$

Thus

$$\varphi_m(z) \ge \frac{1}{m} \log|f(z)| = \varphi(z) - \frac{1}{2m} \log C$$

and $\varphi - \frac{C}{m} \le \varphi_m$. Therefore

$$\nu_{\varphi_m} \le \nu_{\varphi - C/m} = \nu_\varphi.$$

Other inequalities are elementary (they follow from the Poisson representation). $\square$

Demailly's result easily implies

Siu's Theorem (1974). $\varphi \in PSH(\Omega)$, $c \in \mathbb{R}$
$\Rightarrow \{\nu_\varphi \geq c\}$ is an analytic subset of Ω.

Proof (Demailly, 1992): $\{\nu_\varphi \geq c\} = \bigcap_m \{\nu_{\varphi_m} \geq c - \dfrac{n}{m}\}$.

The approximating functions φ_m have only analytic singularities: locally they are of the form

$$\varphi_m = \log(|g_1|^2 + \cdots + |g_k|^2) + u,$$

where $g_1, \ldots, g_k$ are holomorphic and u is C^∞-smooth. Therefore the sets $\{\nu_{\varphi_m} \geq c - \frac{n}{m}\}$ are analytic. $\square$

Siu's theorem is thus a rather simple consequence of the Ohsawa-Takegoshi theorem.

Theorem (Demailly-Peternell-Schneider, 2001)
$\exists\ C = C(\Omega) > 0$ s.th.

$$(m_1 + m_2)\varphi_{m_1+m_2} \leq C + m_1\varphi_{m_1} + m_2\varphi_{m_2}.$$

Proof: O-T from diagonal of $\Omega \times \Omega$ to $\Omega \times \Omega$.

Corollary. The (sub)sequence $\varphi_{2^k} + C/2^{k+1}$ is decreasing.

Open problem: Is the whole sequence φ_m (possibly modified by constants decreasing to 0) decreasing?

D bounded in $\mathbb{C}$ ($n = 1!$)
Logarithmic capacity of D w.r.t. $z \in D$:

$$c_D(z) := \exp \lim_{\zeta \to z} (G_D(\zeta, z) - \log|\zeta - z|),$$

where G_D is the (negative) Green function of D.

Suita Conjecture (1972): $c_D^2 \le \pi K_D$

"$=$" if D is simply connected, "$<$" if D is an annulus (Suita)
Suita also showed that $\pi K_\Omega = \psi_{z\bar{z}}$, where $\psi := \log c_\Omega$ (Robin function). Thus

$$\text{SC} \ \Leftrightarrow \ e^{2\psi} \le \psi_{z\bar{z}} \ \Leftrightarrow \ K_{e^\psi |dz|} \le -4$$

One may assume that D has smooth boundary. Then $K_{e^\psi |dz|} = -4$ on ∂D. Does $K_{e^\psi |dz|}$ satisfy the maximum principle in D?

$$\text{SC} \;\Leftrightarrow\; \forall\, z \in D\; \exists f \in \mathcal{O}(D) : f(z) = 1,\; \int_D |f|^2 d\lambda \le \frac{\pi}{c_D(z)^2}$$

Theorem (Ohsawa, 1995). $c_D^2 \le 750\pi K_D$

Proof: Methods of the original proof of O-T (L^2-theory, commutator identities on Kähler manifolds, etc.)

Theorem (B., 2007). $c_D^2 \le 2\pi K_D$

Proof: $0 \in D$, $G := G_D(\cdot, 0)$

$$\varphi := 2(\log|z| - G)$$

φ is harmonic in D, $c_D(0)^2 = e^{-\varphi(0)}$

We will use the notation $\partial = \dfrac{\partial}{\partial z}, \quad \bar\partial = \dfrac{\partial}{\partial \bar z}$.

$$\bar{\partial}^*\alpha = -e^{\varphi}\partial(e^{-\varphi}\alpha) = -\partial\alpha + \alpha\partial\varphi,$$
$$\Box\alpha = -\bar{\partial}^*\,\bar{\partial}\alpha = \partial\bar{\partial}\alpha - \partial\varphi\bar{\partial}\alpha.$$

$\exists!\ N \in C^\infty(\overline{D}\setminus\{0\}) \cap L^1(D)$ s.th.

$$\Box N = \frac{\pi}{2}e^{\varphi(0)}\delta_0, \quad N = 0 \text{ on } \partial\Omega.$$

One can show that

$$\bar{\partial}(e^{-\varphi}\partial\bar{N}) = \frac{\pi}{2}\delta_0,$$

thus

$$f := ze^{-\varphi}\partial\overline{N} \in \mathcal{O}(D), \quad f(0) = 1/2.$$

One can show that $\int_D |f|^2 d\lambda \le \frac{\pi}{2}e^{\varphi(0)} = \frac{\pi}{2}c_D(0)^{-2}$.
(Main tool (Berndtsson, 1992): $|N|^2 \le e^{\varphi+\varphi(0)}G^2$.) $\Box$

No L^2-theory, only PDE's!

One can combine O-T and Ohsawa's inequality in one result:

Theorem (Ohsawa, 2001, Z.Dinew, 2007).
D bounded domain in $\mathbb{C}$, $0 \in D$,
$\Omega \subset D \times \mathbb{C}^{n-1}$ pseudoconvex, $\varphi \in PSH(\Omega)$,
$H := \{z_1 = 0\}$, $\Omega' := \Omega \cap H$,
$f \in \mathcal{O}(\Omega')$
$\Rightarrow \exists\ F \in \mathcal{O}(\Omega)$ s.th. $F|_{\Omega'} = f$ and

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{4\pi}{c_D(0)^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

Suita Conjecture in SCV: Can one replace 4π with π in the above estimate?

Can one avoid the L^2-theory in the proof of O-T?