Ohsawa-Takegoshi Extension Theorem and Applications

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Theorem (Ohsawa-Takegoshi, 1987).

\[ \Omega \subset \mathbb{C}^n \text{ bounded, pseudoconvex, } \varphi \in PSH(\Omega), \]

\[ H \text{ hyperplane in } \mathbb{C}^n, \Omega' := \Omega \cap H, \]

\[ f \in O(\Omega') \]

\[ \Rightarrow \exists F \in O(\Omega) \text{ s.th. } F_{|\Omega'} = f \text{ and } \]

\[ \int_{\Omega} |F|^2 e^{-\varphi} d\lambda \leq C(n, \text{diam } \Omega) \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'. \]

Original proof: \( L^2 \)-theory of \( \overline{\partial} \)-equation on complete Kähler manifolds + commutator identities (in the spirit of Bochner, Kodaira, Nakano...).

Berndtsson (1996): proof without employing any Kähler metrics; if \( H = \{ z_1 = 0 \} \) and \( \Omega \subset \{ |z_1| < 1 \} \) then \( C = 4\pi \).
Bergman kernel

\[ K_\Omega := \sup\{|f|^2 : f \in \mathcal{O}(\Omega), \int_\Omega |f|^2 d\lambda \leq 1\} \]

(on the diagonal)

O-T implies that

\[ K_{\Omega'} \leq C K_\Omega \text{ on } \Omega'. \]

Corollary (original motivation behind O-T). \( \Omega \) bounded, pseudoconvex, with \( C^2 \) boundary. Then

\[ K_\Omega(z) \geq \frac{1}{C(\text{dist}(z, \partial\Omega))^2}, \quad z \in \Omega, \]

for some \( C = C(\Omega) > 0. \)
\( \varphi \in PSH(\Omega) \), i.e. \( \left( \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \right) \geq 0 \)

Lelong number of \( \varphi \) at \( z_0 \):

\[
\nu_\varphi(z_0) = \lim_{z \to z_0} \frac{\varphi(z)}{\log |z - z_0|} = \lim_{r \to 0^+} \frac{\varphi^r(z_0)}{\log r},
\]

where for \( r > 0 \)

\[
\varphi^r(z) := \max_{\overline{B}(z,r)} \varphi
\]

\( (z \in \Omega_r := \{ z \in \Omega : \text{dist} (z, \partial \Omega) > r \}) \).

One can show that \( \varphi^r \) is a plurisubharmonic continuous function in \( \Omega_r \), decreasing to \( \varphi \) as \( r \) decreases to 0.
Demailly Approximation

Theorem (Demailly, 1992). $\varphi \in PSH(\Omega)$,

$$\varphi_m := \frac{1}{2m} \log K_{\Omega,e^{-2m\varphi}}, \quad m = 1, 2 \ldots,$$

where

$$K_{\Omega,e^{-2m\varphi}} = \sup\{|f|^2 : f \in \mathcal{O}(\Omega), \int_{\Omega} |f|^2 e^{-2m\varphi} \leq 1\}.$$

$\Rightarrow \quad \exists \ C_1, C_2 > 0$ depending only on $\Omega$ s.th.

$$\varphi - \frac{C_1}{m} \leq \varphi_m \leq \varphi^r + \frac{1}{m} \log \frac{C_2}{r^n} \quad \text{in } \Omega_r.$$

In particular, $\varphi_m \rightarrow \varphi$ pointwise and in $L^1_{loc}(\Omega)$.

Moreover,

$$\nu_\varphi - \frac{n}{m} \leq \nu_{\varphi_m} \leq \nu_\varphi \quad \text{in } \Omega.$$
Proof: For a fixed $z \in \Omega$, by O-T (extension from a single point to $\Omega$) one can find $f \in \mathcal{O}(\Omega)$ s.th.

$$\int_{\Omega} |f|^2 e^{-2m\varphi} \, d\lambda \leq C |f(z)|^2 e^{-2m\varphi(z)} = 1.$$ 

Thus

$$\varphi_m(z) \geq \frac{1}{m} \log |f(z)| = \varphi(z) - \frac{1}{2m} \log C$$

and $\varphi - \frac{C}{m} \leq \varphi_m$. Therefore

$$\nu_{\varphi_m} \leq \nu_{\varphi - C/m} = \nu_{\varphi}.$$ 

Other inequalities are elementary (they follow from the Poisson representation). \qed
Demailly’s result easily implies

**Siu’s Theorem (1974).** \( \varphi \in PSH(\Omega), \ c \in \mathbb{R} \) 
\( \Rightarrow \ \{\nu_{\varphi} \geq c\} \) is an analytic subset of \( \Omega \).

**Proof (Demailly, 1992):** \( \{\nu_{\varphi} \geq c\} = \bigcap_{m} \{\nu_{\varphi_m} \geq c - \frac{n}{m}\} \).

The approximating functions \( \varphi_m \) have only analytic singularities: locally they are of the form

\[
\varphi_m = \log(|g_1|^2 + \cdots + |g_k|^2) + u,
\]

where \( g_1, \ldots, g_k \) are holomorphic and \( u \) is \( C^\infty \)-smooth. Therefore the sets \( \{\nu_{\varphi_m} \geq c - \frac{n}{m}\} \) are analytic. \( \square \)

Siu’s theorem is thus a rather simple consequence of the Ohsawa-Takegoshi theorem.
Theorem (Demailly-Peternell-Schneider, 2001)
\[ \exists C = C(\Omega) > 0 \text{ s.th.} \]
\[ (m_1 + m_2) \varphi_{m_1+m_2} \leq C + m_1 \varphi_{m_1} + m_2 \varphi_{m_2}. \]

Proof: O-T from diagonal of \( \Omega \times \Omega \) to \( \Omega \times \Omega \).

Corollary. The (sub)sequence \( \varphi_{2k} + C/2^{k+1} \) is decreasing.

Open problem: Is the whole sequence \( \varphi_m \) (possibly modified by constants decreasing to 0) decreasing?
$D$ bounded in $\mathbb{C}$ ($n = 1$!)

Logarithmic capacity of $D$ w.r.t. $z \in D$:

$$c_D(z) := \exp \lim_{\zeta \to z} (G_D(\zeta, z) - \log |\zeta - z|),$$

where $G_D$ is the (negative) Green function of $D$.

**Suita Conjecture (1972):** $c_D^2 \leq \pi K_D$

"=" if $D$ is simply connected, "<" if $D$ is an annulus (Suita)

Suita also showed that $\pi K_\Omega = \psi_{z\bar{z}}$, where $\psi := \log c_\Omega$ (Robin function). Thus

$$SC \iff e^{2\psi} \leq \psi_{z\bar{z}} \iff K_{e^\psi}|dz| \leq -4$$

One may assume that $D$ has smooth boundary. Then $K_{e^\psi}|dz| = -4$ on $\partial D$. Does $K_{e^\psi}|dz|$ satisfy the maximum principle in $D$?
SC \iff \forall z \in D \exists f \in \mathcal{O}(D) : f(z) = 1, \quad \int_D |f|^2 d\lambda \leq \frac{\pi}{c_D(z)^2}

\textbf{Theorem (Ohsawa, 1995).} \quad c_D^2 \leq 750\pi K_D

\textbf{Proof:} Methods of the original proof of O-T ($L^2$-theory, commutator identities on Kähler manifolds, etc.)

\textbf{Theorem (B., 2007).} \quad c_D^2 \leq 2\pi K_D

\textbf{Proof:} 0 \in D, G := G_D(\cdot, 0)

\varphi := 2(\log |z| - G)

\varphi \text{ is harmonic in } D, \quad c_D(0)^2 = e^{-\varphi(0)}

We will use the notation $\partial = \frac{\partial}{\partial z}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}$. 
\[ \bar{\partial}^* \alpha = -e^\varphi \partial(e^{-\varphi} \alpha) = -\partial \alpha + \alpha \partial \varphi, \]
\[ \Box \alpha = -\bar{\partial}^* \bar{\partial} \alpha = \partial \bar{\partial} \alpha - \partial \varphi \bar{\partial} \alpha. \]

\[ \exists! \ N \in C^\infty(\overline{D} \setminus \{0\}) \cap L^1(D) \text{ s.th.} \]
\[ \Box N = \frac{\pi}{2} e^{\varphi(0)} \delta_0, \quad N = 0 \text{ on } \partial \Omega. \]

One can show that
\[ \bar{\partial}(e^{-\varphi} \partial \bar{N}) = \frac{\pi}{2} \delta_0, \]
thus
\[ f := z e^{-\varphi} \partial \bar{N} \in O(D), \quad f(0) = 1/2. \]

One can show that
\[ \int_D |f|^2 d\lambda \leq \frac{\pi}{2} e^{\varphi(0)} = \frac{\pi}{2} c_D(0)^{-2}. \]
(Main tool (Berndtsson, 1992): \[ |N|^2 \leq e^{\varphi + \varphi(0)} G^2. \])

No \( L^2 \)-theory, only PDE’s!
One can combine O-T and Ohsawa’s inequality in one result:

**Theorem (Ohsawa, 2001, Z.Dinew, 2007).**

\( D \) bounded domain in \( \mathbb{C} \), \( 0 \in D \),
\( \Omega \subset D \times \mathbb{C}^{n-1} \) pseudoconvex, \( \varphi \in PSH(\Omega) \),
\( H := \{z_1 = 0\} \), \( \Omega' := \Omega \cap H \),
\( f \in \mathcal{O}(\Omega') \)

\( \Rightarrow \exists \ F \in \mathcal{O}(\Omega) \) s.th. \( F|_{\Omega'} = f \) and

\[
\int_{\Omega} |F|^2 e^{-\varphi} d\lambda \leq \frac{4\pi}{c_D(0)^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.
\]

**Suita Conjecture in SCV:** Can one replace \( 4\pi \) with \( \pi \) in the above estimate?

Can one avoid the \( L^2 \)-theory in the proof of O-T?