Local Regularity of the Monge-Ampère Equation

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We are interested in local regularity of
(RMA) \[ \det(u_{x_ix_j}) = f > 0 \]
(for convex solutions in domains in \( \mathbb{R}^n \)) and
(CMA) \[ \det(u_{z_i\bar{z}_j}) = f > 0 \]
(for continuous psh solutions in domains in \( \mathbb{C}^n \)).
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   2. Makes no sense to allow \(f \geq 0\).
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2. Makes no sense to allow $f \geq 0$.

Natural question: $f \in C^\infty \Rightarrow u \in C^\infty$?
Real Monge-Ampère Equation
Real Monge-Ampère Equation

Example (Pogorelov, 1971). \( u(x) = (x_1^2 + 1)|x'|^{2\beta}, \beta \geq 0, \) where \( x' = (x_2, \ldots, x_n). \)
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\[
\det(u_{x_ix_j}) = c(1+x_1^2)^{n-2} \left[ (2\beta-1)-(2\beta+1)x_1^2 \right] |x'|^{2(\beta n+1-n)}.
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**Theorem.** \( n = 2, \ f \in C^\infty \Rightarrow u \in C^\infty \)

**Theorem (Aleksandrov, 1942).**
\( n = 2, \ \det(u_{x_i x_j}) \geq c > 0 \Rightarrow u \) is strictly convex

**Theorem (Pogorelov, 1971).**
\( \Omega \subset\subset \mathbb{R}^n, \ u = 0 \text{ on } \partial \Omega, \ f \in C^\infty(\Omega) \Rightarrow u \in C^\infty(\Omega) \)
Coming back to Pogorelov’s example \((n \geq 3)\):

\[
    u = (x_1^2 + 1)|x'|^{2(1 - 1/n)},
\]

so that \(f = c(1 + x_1^2)^{n-2}\).
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$$u = (x_1^2 + 1)|x'|^{2(1-1/n)},$$

so that $f = c(1 + x_1^2)^{n-2}$. Then

$$u \in W^{2,p}_{loc} \iff p < \frac{1}{2}n(n - 1)$$

and

$$u \in C^{1,\alpha} \iff \alpha \leq 1 - \frac{2}{n}.$$
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**Theorem (Urbas, 1988).** If $n \geq 3$ and

- either $u \in W^{2,p}_{loc}$ for some $p > n(n - 1)/2$
- or $u \in C^{1,\alpha}$ for some $\alpha > 1 - 2/n$

then

$$f \in C^\infty \implies u \in C^\infty.$$
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**Example.** $u(z) = (1 + |z_1|^2)|z'|^{2(1-1/n)}$ is psh in $\mathbb{C}^n$, 
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In particular, $u(z_1, z_2) = 2(1 + |z_1|^2)|z_2|$ satisfies $\det(u_{zi \bar{z}_j}) = 1$. 
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\[ u \in W^{2,p}_{loc} \iff p < n(n - 1), \quad u \in C^{1,\alpha} \iff \alpha \leq 1 - \frac{2}{n}. \]
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Theorem (B.-S. Dinew). If \( u \in W^{2,p}_{loc} \) for some \( p > n(n-1) \) then
\[
f \in C^{\infty} \Rightarrow u \in C^{\infty}.
\]
More precisely we have

**Theorem.** Assume \( \Omega \subset \mathbb{C}^n, n \geq 2, \)

\( u \in PSH \cap W^{2,p}(\Omega) \) for some \( p > n(n - 1) \) solves

\[
\det(u_{\bar{z}_i z_j}) = f > 0,
\]

where \( f \in C^{1,1}(\Omega). \)

Then \( \Delta u \in L^\infty_{loc}(\Omega) \) and for \( \Omega' \subset \subset \Omega \)

\[
\sup_{\Omega'} \Delta u \leq C,
\]

where \( C \) depends only on \( n, p, \| f \|_{C^{1,1}(\Omega)}, \inf_{\Omega} f, \| \Delta u \|_{L^p(\Omega)} \) and \( \text{dist}(\Omega', \partial \Omega). \)
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\frac{u^i}{\bar{j}} (\Delta u)_{i\bar{j}} \geq \Delta (\log f) \geq -C_1.
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Set \( w := (1 - |z|^2) \alpha (\Delta u) \beta, \alpha, \beta \geq 2 \).
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$$u^{ij}(\Delta u)_{ij} \geq \Delta (\log f) \geq -C_1.$$ 

Set $w := (1 - |z|^2)^{\alpha}(\Delta u)^{\beta}$, $\alpha, \beta \geq 2$. After some computations we will get

$$u^{ij}w_{ij} \geq -C_2(\Delta u)^{\alpha-1} - C_3w^{1-2/\beta}(\Delta u)^{2\alpha/\beta} \sum_{i,j} |u^{ij}|.$$
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$$\alpha = 1 + \frac{p}{qn}, \quad \beta = 2(1 + \frac{qn}{p}).$$

Then

$$\|(u^{i\bar{j}} w^{i\bar{j}})_{-}\|_{qn} \leq C_4 (1 + (\sup_{B} w)^{1-2/\beta}),$$

where $f_- := -\min(f, 0)$. 
Solve \( \det(v_{ij}) = ((u^{ij}w_{ij})^-)^n \), \( v = 0 \) on \( \partial B \). Then

\[
\sup_B w \leq C_5 \sup_B (-v) \\
\leq C_6 \| \det(v_{ij}) \|_q^{1/n} \\
= C_6 \| (u^{ij}w_{ij})^- \|_{qn} \\
\leq C_7 (1 + (\sup_B w)^{1-2/\beta})
\]

by Kołodziej’s estimate.
Solve $\det(v_{ij}) = ((u^{ij}w_{ij})_-)^n$, $v = 0$ on $\partial B$. Then

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$$= C_6 \| (u^{ij}w_{ij})_- \|_{qn}$$

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by Kołodziej’s estimate. Therefore

$$w = (1 - |z|^2)^\alpha (\Delta u)^\beta \leq C_8.$$
For $u$ which is just in $W^{2,p}$ we consider

$$T = T_\varepsilon u = \frac{n + 1}{\varepsilon^2} (u_\varepsilon - u),$$

where

$$u_\varepsilon(z) = \frac{1}{\lambda(B(z, \varepsilon))} \int_{B(z,\varepsilon)} u \, d\lambda.$$ 

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Then $T_\varepsilon u \to \Delta u$ weakly. One can show that

$$u^{ij}T_{ij} \geq nf^{-1/n}T_\varepsilon(f^{1/n}) \geq -C_9$$

and now we can work as before with $T$ instead of $\Delta u$. 
Theorem (S. Dinew-X. Zhang-X.W. Zhang). \(0 < \alpha < 1\). For \(u \in C^{1,1}\) we have
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It would useful to weaken the assumption to $\Delta u \in L^\infty_{loc}$. 
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It would useful to weaken the assumption to $\Delta u \in L^\infty_{loc}$. For this the following version of Bedford-Taylor’s interior regularity would be sufficient:

Assume $v$ is psh and has bounded Laplacian near $\bar{B}$. Let $u$ be the psh solution of $\det(u_{i\bar{j}}) = 1$, $u = v$ on $\partial B$. Then $\Delta u \in L^\infty_{loc}(B)$. 