Local Regularity of the Monge-Ampère Equation

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We are interested in local regularity of

$$\text{(RMA)} \qquad \det(u_{x_i x_j}) = f > 0$$

(for convex solutions in domains in $\mathbb{R}^n$) and

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2. Makes no sense to allow $f \geq 0$.

Natural question: $f \in C^\infty \;\Rightarrow\; u \in C^\infty$?

Real Monge-Ampère Equation

Example (Pogorelov, 1971). $u(x) = (x_1^2 + 1)|x'|^{2\beta}$, $\beta \geq 0$, where $x' = (x_2, \ldots, x_n)$.

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Theorem (Pogorelov, 1971).
$\Omega \subset\subset \mathbb{R}^n,\ u = 0$ on $\partial\Omega,\ f \in C^\infty(\Omega) \ \Rightarrow\ u \in C^\infty(\Omega)$

Coming back to Pogorelov's example ($n \geq 3$):

$$u = (x_1^2 + 1)|x'|^{2(1-1/n)},$$

so that $f = c(1 + x_1^2)^{n-2}$.

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$$u \in W^{2,p}_{loc} \;\Leftrightarrow\; p < \frac{1}{2}n(n-1)$$

and

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Theorem (Urbas, 1988). If $n \geq 3$ and

- either $u \in W^{2,p}_{loc}$ for some $p > n(n-1)/2$
- or $u \in C^{1,\alpha}$ for some $\alpha > 1 - 2/n$

then

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Theorem (B.-S. Dinew). If $u \in W^{2,p}_{loc}$ for some $p > n(n-1)$ then

$$f \in C^\infty \;\Rightarrow\; u \in C^\infty.$$

More precisely we have

Theorem. Assume $\Omega \subset \mathbb{C}^n$, $n \geq 2$,
$u \in PSH \cap W^{2,p}(\Omega)$ for some $p > n(n-1)$ solves

$$\det(u_{z_i \bar{z}_j}) = f > 0,$$

where $f \in C^{1,1}(\Omega)$.
Then $\Delta u \in L^\infty_{loc}(\Omega)$ and for $\Omega' \subset\subset \Omega$

$$\sup_{\Omega'} \Delta u \leq C,$$

where C depends only on n, p, $||f||_{C^{1,1}(\Omega)}$, $\inf_\Omega f$, $||\Delta u||_{L^p(\Omega)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$.

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$$u^{i\bar{j}} w_{i\bar{j}} \geq -C_2(\Delta u)^{\alpha-1} - C_3 w^{1-2/\beta}(\Delta u)^{2\alpha/\beta} \sum_{i,j} |u^{i\bar{j}}|.$$

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Fix $1 < q < p/(n(n-1))$. Since $||\Delta u||_p$ is under control, it follows that $||u_{i\bar{j}}||_p$ and $||u^{i\bar{j}}||_{p/(n-1)}$ are as well.

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$$\alpha = 1 + \frac{p}{qn}, \quad \beta = 2\big(1 + \frac{qn}{p}\big).$$

Then

$$||(u^{i\bar{j}} w_{i\bar{j}})_-||_{qn} \leq C_4(1 + (\sup_B w)^{1-2/\beta}),$$

where $f_- := -\min(f, 0)$.

Solve $\det(v_{i\bar{j}}) = ((u^{i\bar{j}} w_{i\bar{j}})_-)^n$, $v = 0$ on ∂B. Then

$$\begin{aligned}\sup_B w &\leq C_5 \sup_B(-v) \\ &\leq C_6 || \det(v_{i\bar{j}}) ||_q^{1/n} \\ &= C_6 || (u^{i\bar{j}} w_{i\bar{j}})_- ||_{qn} \\ &\leq C_7 (1 + (\sup_B \, w)^{1-2/\beta})\end{aligned}$$

by Kołodziej's estimate.

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$$w = (1 - |z|^2)^\alpha (\Delta u)^\beta \le C_8.$$

For u which is just in $W^{2,p}$ we consider

$$T = T_\varepsilon u = \frac{n+1}{\varepsilon^2}(u_\varepsilon - u),$$

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$$u_\varepsilon(z) = \frac{1}{\lambda(B(z,\varepsilon))} \int_{B(z,\varepsilon)} u \, d\lambda.$$

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Then $T_\varepsilon u \to \Delta u$ weakly. One can show that

$$u^{i\bar{j}} T_{i\bar{j}} \geq n f^{-1/n} T_\varepsilon(f^{1/n}) \geq -C_9$$

and now we can work as before with T instead of Δu.

Theorem (S. Dinew-X. Zhang-X.W. Zhang). $0 < \alpha < 1$.
For $u \in C^{1,1}$ we have

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It would useful to weaken the assumption to $\Delta u \in L^{\infty}_{loc}$.
For this the following version of Bedford-Taylor's interior regularity would be sufficient:

Assume v is psh and has bounded Laplacian near $\bar{B}$. Let u be the psh solution of $\det(u_{i\bar{j}}) = 1$, $u = v$ on ∂B.
Then $\Delta u \in L^{\infty}_{loc}(B)$.