Conference on Several Complex Variables on the occasion of Professor Józef Siciak's 80th birthday

Kraków, July 4–8, 2011

Abstracts of invited talks

Bo Berndtsson (Göteborg)

Variants of the Moser-Trudinger inequality

The classical Moser-Trudinger inequality is an estimate for the integral of the exponential of a function over the Riemann sphere in terms of the L^2-norm of its gradient. Generalizations of this to Fano manifolds possessing a Kähler-Einstein metric play an important role in Kähler geometry. We will discuss variants of this for more general complex manifolds and also for domains in C^n.

This is joint work with Robert Berman.

Thomas Bloom (Toronto)

Multiple orthogonal polynomials

Multiple orthogonal polynomials (MOP) are an extension of orthogonal polynomials. Probability measures (ensembles) associated to MOP's arise, for example, in some aspects of Brownian motion.

We will show how potential theory can be used to establish almost sure convergence of the normalized counting measure of a random point for certain MOP ensembles.

Bogdan Bojarski (IM PAN)

Taylor expansion and Sobolev spaces

A new characterisation of functions in Sobolev spaces $W^{m,p}(\mathbb{R}^n)$, $m > 1$, in the form of a pointwise inequality is discussed. It reveals the local and global polynomial-like behaviour of these functions.

Len Bos (Verona)

Polynomial interpolation at points in the real disk

We discuss a bivariate polynomial interpolation scheme for the real disk, based on points situated on concentric circles. Of special interest is their relation to the equilibrium measure and transfinite diameter.

Jean-Paul Calvi (Toulouse)

Extremal points for polynomial interpolation: the constructive approach

We discuss recent advances on the construction and the computation of good interpolation points for univariate and multivariate Lagrange interpolation.

Pierre Dolbeault (Paris)

Boundaries of Levi-flat hypersurfaces: special hyperbolic points

Let $S \subset \mathbb{C}^n$, $n \geq 3$, be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface whose boundary is S, possibly as a current. Our goal is to get examples of such S containing at least one special 1-hyperbolic point: sphere with two horns; elementary models and their gluing.

Franc Forstnerič (Ljubljana)
The Poletsky-Rosay theorem on singular complex spaces

If u is an upper semicontinuous function on a locally irreducible complex space X, then the largest plurisubharmonic function v that is less or equal to u is obtained as the pointwise infimum of the averages of u over the boundaries of analytic discs in X. This was proved by Poletsky (1993) for $X = \mathbb{C}^n$ and by Rosay (2003) for X a complex manifold. Applications include the description of the plurisubharmonic and the pluripolar hull of a compact set in a complex space.

Josip Globevnik (Ljubljana)
Small families of complex lines for testing holomorphic extendibility from spheres

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^2$. Let f be a continuous function on $b\mathbb{B}$. If L is a complex line that meets $\mathbb{B}$ then we say that the function f extends holomorphically into $\mathbb{B}$ along L if $f|_{L\cap b\mathbb{B}}$ extends holomorphically through $L \cap \mathbb{B}$. We consider the question about along how many complex lines should f extend holomorphically into $\mathbb{B}$ in order that f extends holomorphically through $\mathbb{B}$. Denote by $\mathcal{L}(a)$ the set of all complex lines passing through a.

Theorem 1. *Let a, b be two points in $\mathbb{C}^2$ such that the complex line through a and b meets $\mathbb{B}$ and such that $\langle a|b\rangle \neq 1$ if one of the points is contained in $\mathbb{B}$ and the other in $\mathbb{C}^2 \setminus \overline{\mathbb{B}}$. If a function $f \in \mathcal{C}^\infty(b\mathbb{B})$ extends holomorphically into $\mathbb{B}$ along each $L \in \mathcal{L}(a) \cup \mathcal{L}(b)$ then f extends holomorphically through $\mathbb{B}$.*

When $a, b \in \overline{\mathbb{B}}$ and when f is real analytic, this theorem was proved by M. Agranovsky. Such a theorem fails to hold for functions in $\mathcal{C}^k(b\mathbb{B})$.

Theorem 2. *Let a, b, c be three points in $\mathbb{C}^2$ which do not lie in a complex line, such that the complex line through a, b meets $\mathbb{B}$ and such that if one of the points a, b is in $\mathbb{B}$ and the other in $\mathbb{C}^2 \setminus \overline{\mathbb{B}}$ then $\langle a|b\rangle \neq 1$ and such that at least one of the numbers $\langle a|c\rangle,\ \langle b|c\rangle$ is different from 1. If a continuous function f on $b\mathbb{B}$ extends holomorphically into $\mathbb{B}$ along each $L \in \mathcal{L}(a) \cup \mathcal{L}(b) \cup \mathcal{L}(c)$ then f extends holomorphically through $\mathbb{B}$.*

In the special case when a, b, c are contained in $\mathbb{B}$ this theorem was proved by L. Baracco.
In the talk we indicate how to prove these theorems and on the way we present two new one variable results.

Norman Levenberg (Bloomington)
The maximum modulus principle and projective hulls

Let Ω be an open subset of the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and let $T = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle. Let $\mathcal{M}$ be a vector space of complex-valued, continuous functions on $T \cup \Omega$ containing the constants and closed under multiplication by z. Hence $\mathcal{M}$ contains all holomorphic polynomials. If the functions in $\mathcal{M}$ satisfy a version of the maximum modulus principle, then one should be able to deduce analyticity of these functions in Ω. For example, one may consider a weak version of this principle:

For all $z_0 \in \Omega$ there exists C_{z_0} such that $|f(z_0)| \leq C_{z_0}\|f\|_T$ for all $f \in \mathcal{M}$.

Suppose $\Omega = \Delta$. *When can one conclude that each $f \in \mathcal{M}$ is holomorphic in Δ? Suppose* $\Omega = \Delta \setminus \{0\}$. *When can one conclude that each $f \in \mathcal{M}$ is meromorphic in Δ?* In this latter setting we get into issues involving projective hulls of sets in $\mathbb{C}^2$. We discuss some recent results and conjectures of John Wermer—see his preprint *Rudin's Theorem and Projective Hulls* on the arXiv—as well as our contributions and dreams.
This is joint work with John Anderson, Joe Cima and Tom Ransford.

Takeo Ohsawa (Nagoya)
Hartogs type extension theorems on some domains in Kähler manifolds

It was recently shown that the boundaries of bounded pseudoconvex domains with $\mathcal{C}^2$-smooth boundary in Kähler manifolds are connected unless they are Levi flat. The proof is based on a Hartogs type extension theorem. The proof by L^2 cohomology vanishing will be sketched and an extension of the method will be discussed.

Evgeny Poletsky (Syracuse)
Padé interpolation by $\boldsymbol{F}$-polynomials and transfinite diameter

We define F-polynomials as linear combinations of dilations of an entire functions F. We use Padé interpolation by F-polynomials to obtain explicitly approximating F-polynomials with estimates for their coefficients. We show that when frequencies lies in a compact set K in the complex plane then optimal choices for the frequencies of interpolating polynomials are similar to Fekete points and the minimal norms of the interpolating operators form a sequence whose rate of growth is determined by the transfinite diameter.
For the Laplace transforms of measures on K we show that the coefficients of interpolating polynomials stay bounded provided the frequencies are Fekete points. Finally, we give sufficient condition for measures on the unit circle so that the sums of the absolute values of the coefficients of interpolating polynomials stay bounded.
This is joint work with Dan Coman.

Azimbay Sadullaev (Tashkent)
History of pluripotential theory and some its applications in the multidimensional complex analysis

Nessim Sibony (Paris)
On some recent results in pluripotential theory and dynamics

Pluripotential has been an important tool in the recent development of holomorphic dynamics in several complex variables. I will illustrate this usefulness with some recent results in dynamics. In particular, I will use the space of (p,p) forms whose dd^c is a difference of two positive closed currents.

Nikolay Shcherbina (Wuppertal)
On defining functions for unbounded pseudoconvex domains

We show that there exist an unbounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and a Wemer type set $\mathcal{E} \subset \Omega$ such that each defining function for Ω fails to be strictly plurisubharmonic on $\mathcal{E}$. We also prove a Liouville theorem for Wermer type sets.

Jan Wiegerinck (Amsterdam)
Plurifinely plurisubharmonic and holomorphic functions

As is well known, the fine toplogy is the weakest topology on domains in $\mathbb{R}^n$ that makes all subharmonic functions continuous. It allows naturally for finely subharmonic—and in the 2-dimensional case also finely holomorphic—functions.
In $\mathbb{C}^n$ the plurifine topology, which makes all plurisubharmonic functions continuous, is challenging. In this setting we introduce a weak and a strong concept of plurifinely plurisubharmonic and plurifinely holomorphic functions. Strong will imply weak, but it is unknown whether the two concepts are the same.
In this lecture we will discuss the plurifine topology and present our results on plurifinely plurisubharmonic and holomorphic functions.
All this is joint work, partly with Said El Marzguioui, and partly with Mohamed El Kadiri and Bent Fuglede, and it includes

- Every bounded finely plurisubharmonic function can be locally (in the plurifine topology) written as the difference of two usual plurisubharmonic functions. As a consequence finely plurisubharmonic functions are continuous with respect to the plurifine topology.
- The $-\infty$ sets of finely plurisubharmonic functions are pluripolar, hence graphs of finely holomorphic functions are pluripolar.
- A function f is weakly plurifinely plurisubharmonic if and only if it is locally bounded from above in the plurifine topology and $f \circ h$ is finely subharmonic for all complex affine-linear maps h.
- Weak plurifine plurisubharmonicity and weak plurifine holomorphy are preserved under composition with weakly plurifinely holomorphic maps.

Vyacheslav Zakharyuta (Istanbul)
Characteristics of compacta in $\mathbb{C}^n$

The famous classical result of the geometric complex analysis (Fekete, Szegö) states that three characteristics of a compact set $K \subset \mathbb{C}$, which are defined from quite different reasons, do coincide. These characteristics are: *transfinite diameter* $d(K)$, characterising an asymptotic size of K (a geometric approach); *Chebyshev constant* $\tau(K)$, characterising the minimal uniform deviation of monic polynomials on K (an approximation theory approach); *capacity* $c(K)$, describing the asymptotic behavior of the Green function $g_K(z)$ at infinity (a potential theory approach).
The starting points for studying multidimensional analogs of this classical result were: *Leja's transfinite diameter* $d(K)$ for a compact set $K \subset \mathbb{C}^n$ ([2]); the notions of *pluripotential Green function*, *Robin function and capacities in* $\mathbb{C}^n$ ([6], [7], [9], [5], [1], etc.); the concepts of the *directional Chebyshev constants* $\tau(K, \vartheta)$ and *principal Chebyshev constant* $\tau(K)$ and the *equality* $\tau(K) = d(K)$ in $\mathbb{C}^n$ [8]. Recent remarkable results of Rumely and his collaborators showed that multivariate complex analytic methods, developed in connection with the above classical result, have found applications in the arithmetic geometry (see, e.g. [4]). Moreover, Rumely, using methods of arithmetic intersection theory, produced a multidimensional analog of the equality $d(K) = c(K)$ in $\mathbb{C}^n$ [3].
The aim of my talk is to cast a glance to the impressive achievements concerned with the above topics in the last decades (Siciak, Sadullaev, Zeriahi, Bloom, Levenberg, Calvi, Jędrzdejowski, Kołodziej, Szczepański et al.) and represent some fresh results based on a new approach to the transfinite diameter and related notions.

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Lawrence Zalcman (Ramat Gan)
100 years of normal families

We survey the theory of normal families of meromorphic functions on plane domains, with an emphasis on some remarkable developments of the past third of a century.