ON THE POINTWISE OHSAWA-TAKEGOSHI THEOREM

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Dedicated to the memory of Marek Jarnicki

ABSTRACT. For plurisubharmonic φ in a bounded pseudoconvex Ω in $\mathbb{C}^n$ we analyze the lower bound for the Bergman kernel (on the diagonal, with weight $e^{-\varphi}$) of the form $K_{\Omega,\varphi} \geq e^{\varphi}/C$. By the Ohsawa-Takegoshi extension theorem there exists such a constant depending on n and the diameter of Ω. We give a proof of this using Hörmander's L^2-estimate for $\bar\partial$ directly. We also show that one can improve the constant to $C = |\Omega|$ if $n = 1$ and to $C = 2\,|\Omega|$ if Ω is convex, where $|\Omega|$ denotes the volume (or area) of Ω. We conjecture that the former holds in arbitrary dimension (it trivially does in the unweighted case).

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$ and φ a plurisubharmonic (psh) function in Ω. For $z \in \Omega$ the weighted Bergman kernel (on the diagonal) is defined by

$$K_{\Omega,\varphi}(z) := \sup\left\{\frac{|f(z)|^2}{||f||_\varphi^2} \colon f \in \mathcal{O}(\Omega),\ f \not\equiv 0\right\}. \tag{1}$$

where

$$||f||_\varphi = \sqrt{\int_\Omega |f|^2 e^{-\varphi} d\lambda}.$$

Using standard arguments one can show that $K_{\Omega,\varphi}$ is real analytic in Ω, the supremum in (1) is in fact a maximum (if the weighted Bergman space is not empty) and that if $\Omega_j \nearrow \Omega$ and $\varphi_j \searrow \varphi$ then $K_{\Omega_j,\varphi_j} \searrow K_{\Omega,\varphi}$.

If Ω is pseudoconvex then iterating the Ohsawa-Takegoshi theorem [18] on extending holomorphic functions from hyperplanes with L^2-estimate gives the following lower bound

$$K_{\Omega,\varphi} \geq \frac{e^{\varphi}}{C}, \tag{2}$$

where C is a constant depending on $\operatorname{diam}\Omega$ and n. This result is obvious in the unweighted case $\varphi \equiv 0$ but highly non-trivial in general. For example, it easily implies the result of Bombieri [8] (see also [7] and [15]) that for a psh φ the set where $e^{-\varphi}$ is

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not locally integrable is analytic. Also, as proved by Demailly [9], using the pointwise Ohsawa-Takegoshi estimate (2) the sequence of psh functions with analytic singularities

$$\varphi_m := \frac{1}{m} \log K_{\Omega, m\varphi}$$

converges to φ pointwise and in L^1_{loc} as $m \to \infty$ (see [11] for a partial result obtained independently without using Ohsawa-Takegoshi). This approach gives a surprisingly simple proof of the theorem of Siu [20] (see also [15]) on analyticity of superlevel sets for Lelong numbers, see Demailly [9].

We are interested in improving (2) using pluripotential theory. We will use the pluricomplex Green function

$$G_\Omega(z, w) := \sup_{u \in \mathcal{B}_{\Omega,w}} u(z),$$

where $\mathcal{B}_{\Omega,w}$ is the class of negative psh functions in Ω with logarithmic pole at w, that is near w

$$u \leq \log|z - w| + C$$

for some constant C (depending on Ω). It is well known that if $\mathcal{B}_{\Omega,w} \neq \emptyset$ (for example when Ω is bounded) then $G_\Omega(\cdot, w) \in \mathcal{B}_{\Omega,w}$.

The following lower bound was proved in [5] for $\varphi \equiv 0$:

Theorem 1. *Assume that Ω is a pseudoconvex domain in $\mathbb{C}^n$ containing the origin and let $G := G_\Omega(\cdot, 0)$ be the pluricomplex Green function for Ω with pole at the origin. Then for psh φ in Ω and $t \leq 0$ we have*

$$K_{\Omega,\varphi}(0) \geq \frac{e^{2nt}}{\displaystyle\int_{\{G<t\}} e^{-\varphi} d\lambda}. \tag{3}$$

First note that since $G \leq \log(|z|/R)$, where $R = \operatorname{diam}\Omega/2$, we get (2) with

$$C = |B(0, R)| = \frac{\pi^n (\operatorname{diam}\Omega)^{2n}}{4^n n!}.$$

This is the best possible constant depending on $\operatorname{diam}\Omega$, since when Ω is a ball and $\varphi \equiv 0$ then at the center we have equality in (2) with this C.

We will present a new proof of Theorem 1 in Section 2. Compared to the proof from [5] where the Donnelly-Fefferman $\bar\partial$-estimate [10] was used, here we manage by applying the Hörmander estimate directly. But the general procedure is similar: we first prove a weaker version of (3) with a non-optimal constant and then use the tensor power trick to obtain the estimate with the optimal constant e^{2nt}.

It should be pointed out that another, in a way simpler proof of Theorem 1 was found by Lempert [16], see also [2]. Using log-plurisubharmonicity of sections of the Bergman kernel for pseudoconvex domains in $\mathbb{C}^{n+1}$, proved by Maitani-Yamaguchi [17] for $n = 1$ and Berndtsson [1] for arbitary n, he showed that $\log K_{\{G<t\},\varphi}(0)$ is convex

in t, hence $e^{2nt}K_{\{G<t\},\varphi}(0)$ is non-decreasing in t. The latter function is then clearly bounded below by the right-hand side of (3). On the other hand, the proof presented here seems to be more direct having just the Hörmander estimate at our disposal.

When $n = 1$ and D is a domain in $\mathbb{C}$ containing the origin then one can easily show that

$$\lim_{t\to-\infty} e^{-2t}|\{G_D(\cdot,0) < t\}| = \frac{\pi}{(c_D(0))^2}, \tag{4}$$

where

$$c_D(z) = \exp\big(\lim_{\zeta\to z}\big(G_D(\zeta,z) - \log|\zeta - z|\big)\big)$$

is the logarithmic capacity of the complement of D with respect to z. Note that

$$c_D(z) = c(F(\widehat{\mathbb{C}} \setminus D)), \tag{5}$$

where $F(\zeta) = 1/(\zeta - z)$, $\widehat{\mathbb{C}}$ is the Riemann sphere and $c(K)$ is the logarithmic capacity for compact subsets of $\mathbb{C}$. By approximation Theorem 1 gives therefore for any φ subharmonic in D

$$K_{D,\varphi} \geq \frac{1}{\pi}e^{\varphi}c_D^2, \tag{6}$$

a result originally obtained by different methods in [4], for $\varphi \equiv 0$ gives the affirmative answer to the Suita conjecture [22].

Using the isoperimetric inequality it was shown in [6] that the limit in (4) is non-decreasing. This immediately gives

$$c_D \geq \sqrt{\pi/|D|} \tag{7}$$

which in view of (5) is equivalent to the following:

Proposition 2. *If K is a compact in $\mathbb{C}$ and $F(\zeta) = 1/\zeta$ then*

$$c(K) \geq \sqrt{\frac{\pi}{|F(\widehat{\mathbb{C}} \setminus K)|}}. \tag{8}$$

□

The Ahlfors-Beurling inequality (see e.g. [19]) gives

$$c(K) \geq \sqrt{|K|/\pi} \tag{9}$$

(this follows from a similar lower bound for the analytic capacity of K). It would be interesting to verify whether (9) implies (8). It would be the case if the following elementary estimate was true: for any compact K in $\mathbb{C}$ one has

$$|K|\,|\widehat{\mathbb{C}} \setminus F(K)| \geq \pi^2.$$

Equivalently, for any bounded domain D in $\mathbb{C}$ containing the origin one should have

$$|D| \int_{\mathbb{C}\setminus D} \frac{d\lambda}{|\zeta|^4} \geq \pi^2. \tag{10}$$

This would give another proof of (8). It should be pointed out that the proof of (9) is completely different from the proof of (8) using the isoperimetric inequality presented here.

Combining (6) with (7) we get

Corollary 3. *If D is a domain in $\mathbb{C}$ and φ is subharmonic in $\mathbb{C}$ then*

$$K_{D,\varphi} \geq \frac{e^{\varphi}}{|D|}. \tag{11}$$

□

Note that for $\varphi \equiv 0$ this follows trivially from (1), just take $f \equiv 1$ there.

We conjecture that analogous result to Corollary 3 holds for pseudoconvex domains in $\mathbb{C}^n$. It would follow from a still open conjecture posed in [6] saying that if Ω and G are as in Theorem 1 then $e^{-2nt}|\{G < t\}|$ is non-decreasing for $t < 0$. This conjecture is equivalent to the following isoperimetric inequality: if in addition Ω is smoothly bounded and strongly pseudoconvex then

$$\int_{\partial\Omega} \frac{d\sigma}{|\nabla G|} \geq 2n|\Omega| \tag{12}$$

(note that by [3] in this case G is $C^{1,1}$ on $\overline{\Omega} \setminus \{w\}$).

Here we prove the following weaker result for convex domains:

Theorem 4. *Assume that Ω is a convex domain in $\mathbb{C}^n$ containing the origin and $G = G_\Omega(\cdot, 0)$. Then $e^{-nt}|\{G < t\}|$ is non-decreasing for $t < 0$. If Ω is in addition smoothly bounded then*

$$\int_{\partial\Omega} \frac{d\sigma}{|\nabla G|} \geq n|\Omega|. \tag{13}$$

For arbitrary n, as proved in [6] using [24], for sufficiently regular Ω (e.g. bounded and hyperconvex) we have

$$\lim_{t\to-\infty} e^{-2nt}|\{G_\Omega(\cdot, 0) < t\}| = |I_\Omega^A|,$$

where

$$I_\Omega^A = \{v \in \mathbb{C}^n \colon \limsup_{\zeta\to 0} \big(G_\Omega(\zeta v, 0) - \log|\zeta|\big) \leq 0\}$$

is the Azukawa indicatrix of Ω at the origin. By approximation using Theorem 1 we get for arbitrary psh φ in pseudoconvex Ω

$$K_{\Omega,\varphi}(0) \geq \frac{e^{\varphi(0)}}{|I_\Omega^A|}.$$

We conjecture that for pseudoconvex Ω one has

$$|I_\Omega^A| \leq |\Omega|,$$

this would also give Corollary 3 in higher dimensions.

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2. Proof of Theorem 1

Proof of Theorem 1. Without loss of generality we may assume that Ω is bounded and φ is smooth. Define

$$\alpha := \bar\partial(\chi \circ G) = \chi' \circ G\, \bar\partial G \tag{14}$$

where $\chi \in C^{0,1}(\mathbb{R}_-)$ will be determined later. The following elementary result gives a sufficient condition on χ to ensure that $\alpha \in L^2_{loc,(0,1)}(\Omega)$:

Lemma 5. *If u is a smooth negative subharmonic function in an open Ω in $\mathbb{R}^k$ then for $K \Subset \Omega$ and $\chi \in C^{0,1}(\mathbb{R}_-)$ one has*

$$\int_K |\nabla(\chi \circ u)|^2 d\lambda \le C ||u||_{L^1(\Omega)} \int_{-\infty}^0 (\chi')^2 dt, \tag{15}$$

where C depends only on K and Ω. In particular, if u is subharmonic in Ω and locally bounded away from a compact subset of Ω then $u \in W^{1,2}_{loc}(\Omega)$.

Proof. For $t < 0$ set

$$\gamma(t) := \int_{-\infty}^t (\chi'(s))^2 ds, \quad \eta(t) = \int_t^0 \gamma(s) ds,$$

so that $\gamma' = (\chi')^2$ and $\eta' = -\gamma$. First note that by the Fubini theorem

$$\eta(t) = |t| \int_{-\infty}^t (\chi'(s))^2 ds + \int_t^0 |s|(\chi'(s))^2 ds \le |t| \int_{-\infty}^0 (\chi'(s))^2 ds. \tag{16}$$

Let $\phi \in C_0^\infty(\Omega)$ be nonnegative and such that $\phi = 1$ on K. Then

$$\begin{aligned}
\int_K |\nabla(\chi \circ u)|^2 d\lambda &\le \int_\Omega \phi\, (\chi' \circ u)^2\, |\nabla u|^2 d\lambda \\
&= \int_\Omega \phi\, \langle \nabla(\gamma \circ u), \nabla u \rangle\, d\lambda \\
&= -\int_\Omega \phi\, \gamma \circ u\, \Delta u\, d\lambda - \int_\Omega \gamma \circ u\, \langle \nabla \phi, \nabla u \rangle\, d\lambda \\
&\le -\int_\Omega \gamma \circ u\, \langle \nabla \phi, \nabla u \rangle\, d\lambda \\
&= \int_\Omega \langle \nabla \phi, \nabla(\eta \circ u) \rangle\, d\lambda \\
&= -\int_\Omega \eta \circ u\, \Delta \phi\, d\lambda
\end{aligned}$$

and (15) follows from (16). □

Proof of Theorem 1, cont'd. We will use the Hörmander L^2-estimate with α given by (14) and the weight

$$\psi = 2nG - \log(1-G) + \varphi.$$

We have

$$i\bar{\alpha}\wedge\alpha \le (\chi'\circ G)^2(1-G)^2 i\partial\bar\partial\psi.$$

Since $\psi \le \varphi$, we obtain $u \in L^2_{loc}(\Omega)$ solving $\bar\partial u = \alpha$ and such that

$$||u||^2_\varphi \le ||u||^2_\psi \le \int_\Omega (\chi'\circ G)^2(1-G)^3 e^{-2nG}e^{-\varphi}d\lambda. \tag{17}$$

We now choose χ in such a way that the expression depending on G under the integral is equal to the characteristic function of $\{G<t\}$, that is

$$\chi(s) := \begin{cases} \displaystyle\int_s^t \frac{e^{nx}dx}{(1-x)^{3/2}}, & s<t \\ 0, & s \ge t. \end{cases}$$

Then $f := \chi\circ G - u$ is holomorphic in Ω. Since $e^{-\psi}$ is not integrable near the origin (this is because Ω and φ are bounded), we have

$$f(0) = \chi(-\infty) = \int_{-\infty}^t \frac{e^{nx}dx}{(1-x)^{3/2}}. \tag{18}$$

By (17)

$$||f||_\varphi \le ||\chi\circ G||_\varphi + ||u||_\varphi \le \big(\chi(-\infty)+1\big)\sqrt{\int_{\{G<t\}} e^{-\varphi}d\lambda},$$

hence

$$K_{\Omega,\varphi}(0) \ge \frac{|f(0)|^2}{||f||^2_\varphi} \ge \frac{c(n,t)}{\displaystyle\int_{\{G<t\}} e^{-\varphi}d\lambda}, \tag{19}$$

where

$$c(n,t) = \left(\frac{\chi(-\infty)}{\chi(-\infty)+1}\right)^2$$

and $\chi(-\infty)$ is given by (18).

We have thus obtained a weaker version of (3). To get the optimal constant we use the tensor power trick. For $m \gg 0$ consider the product domain $\widetilde\Omega := \Omega^m \subset \mathbb{C}^{nm}$ and

$$\widetilde\varphi(z^1,\dots,z^m) = \varphi(z^1)+\dots+\varphi(z^m).$$

For $\widetilde G = G_{\widetilde\Omega}(\cdot,0)$ we clearly have

$$\widetilde G(z^1,\dots,z^m) \ge \max\{G(z^1),\dots,G(z^m)\}, \tag{20}$$

hence

$$\{\widetilde{G} < t\} \subset \{G < t\}^m.$$

Applying (19) to $\widetilde{\Omega}$ we thus get

$$K_{\Omega,\varphi}(0) \geq \frac{c(nm,t)^{1/m}}{\displaystyle\int_{\{G<t\}} e^{-\varphi} d\lambda}$$

and it is enough to use the fact

$$\lim_{m\to\infty} c(nm,t)^{1/m} = e^{2nt}.$$

This finishes the proof of Theorem 1. □

Remark. In fact, we have equality in the product property (20). This was proved by Jarnicki-Pflug [13] for pseudoconvex domains using the following result of Stoll [21] (see also [23]): if $u(z')$ and $v(z'')$ are locally bounded maximal psh functions (or, equivalently, solving the homogeneous complex Monge-Ampère equation) then so is $\max\{u(z'), v(z'')\}$. For arbitrary, not necessarily psudoconvex domains, it was shown by Edigarian [11] using complex disks methods.

3. Proof of Theorem 4

Let us first recall the argument from [6] that if Ω is pseudoconvex and sufficiently regular then monotonicity of $e^{-2nt}|\{G < t\}|$ is equivalent to (12). In fact, this follows immediately from the fact that if

$$h(t) := \log|\{G < t\}| - 2nt$$

then by the co-area formula, if t is a regular value of G, we have

$$\frac{d}{dt}\lambda(\{G < t\}) = \int_{\{G=t\}} \frac{d\sigma}{|\nabla G|}.$$

It is now also clear that (13) implies monotonicity of $e^{-nt}|\{G < t\}|$.

Theorem 4 will easily follow from the following estimate (it is motivated by the proof of Theorem 5 in [6]):

Lemma 6. *Assume that Ω is bounded convex domain in $\mathbb{C}^n$ containing the origin and such that $\partial\Omega$ is C^1. Denote $G = G_\Omega(\cdot, 0)$. Then for $w \in \partial\Omega$ we have*

$$|\nabla G(w)| \leq \frac{2}{\langle w, n_w\rangle},$$

where n_w is the unit outer normal.

Proof. Let H be the complex tangent space to $\partial\Omega$ at w. We may assume that $H = \{z_1 = w_1\}$. By Ω' denote the projection of Ω along H to $\{z_2 = \dots = z_n = 0\}$. Let l be the real line in $\mathbb{C}$ tangent to $\partial\Omega'$ at w_1 (so that $l \times H$ is the real tangent space to $\partial\Omega$ at w). If $\zeta_0 \in l$ is the shortest point to 0 then

$$u(z) := \log\left|\frac{z_1}{z_1 - 2\zeta_0}\right| \le G(z).$$

Then

$$|\nabla G(w)| \le |\nabla u(w)| = \frac{2}{|\zeta_0|} = \frac{2}{\langle w, n_w\rangle}. \quad \Box$$

$\Box$

Remark. The constant 2 in Lemma 6 cannot be improved: consider a disk in $\mathbb{C}$ and a pole approaching the boundary. $\Box$

Proof of Theorem 4. Using Lemma 6 and integrating by parts we obtain

$$\int_{\partial\Omega} \frac{d\sigma}{|\nabla G|} \ge \frac{1}{2}\int_{\partial\Omega} \langle w, n_w\rangle \, d\sigma(w) = n|\Omega|. \quad \Box$$

$\Box$

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