ON THE DIEDERICH-FORNÆSS EXPONENT

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Abstract. We prove a quantitative lower bound for the Diederich-Fornæss index for bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary. Using pluripotential theory we also characterize the Diederich-Fornæss exponents for possibly nonsmooth plurisubharmonic defining functions in worm domains, generalizing a result of B. Liu.

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. We say that ρ is its defining function if $\rho \in C^2(\overline{\Omega})$, $\rho < 0$ in Ω and $\rho = 0$, $\nabla\rho \neq 0$ on $\partial\Omega$. It is well known that Ω is pseudoconvex if and only if

$$\sum_{j,k} \rho_{z_j\bar{z}_k}(z) w_j \bar{w}_k \geq 0 \tag{1}$$

for all $z \in \partial\Omega$ and $w \in \mathbb{C}^n$ satisfying

$$\sum_j \rho_{z_j}(z) w_j = 0.$$

For $n = 2$ this is equivalent to the following inequality on $\partial\Omega$

$$\rho_{z_1\bar{z}_1}|\rho_{z_2}|^2 - 2\mathrm{Re}\left(\rho_{z_1\bar{z}_2}\rho_{\bar{z}_1}\rho_{z_2}\right) + \rho_{z_2\bar{z}_2}|\rho_{z_1}|^2 \geq 0.$$

Of course, if a defining function is plurisubhmarmonic (psh) in Ω (that is (1) holds for all $z \in \Omega$ and $w \in \mathbb{C}^n$) then Ω is pseudoconvex. The converse however does not hold in general. This was demonstrated by the famous example of the worm domain constructed by Diederich-Fornæss [3]. This kind of phenomenon seems to be very special for several complex variables: it is known that it does not appear neither in convex analysis (see e.g. [7]) nor in the theory of p-convex domains, as recently shown in [5]. On the other hand, Diederich-Fornæss [2] proved that for any bounded pseudoconvex domain with C^2 boundary there exists a defining $\widetilde{\rho}$ and $b > 0$ such that $-(-\widetilde{\rho})^b$ is psh in Ω. Such a b is called the *Diederich-Fornæss (DF) exponent* and their supremum the *DF index* of Ω. It will be denoted by $\eta_2(\Omega)$. If we restrict ourselves to domains and defining functions of class C^k, $k = 2, 3, \ldots, \infty$, then $\eta_k(\Omega)$ will be the corresponding DF index.

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One of the key steps in the proof of this result in [2] was the estimate

$$\delta_{w\bar{w}}(z) \leq C|\delta_w(z)||w|, \quad w \in \mathbb{C}^n, \tag{2}$$

for $z \in \overline{\Omega}$ near $\partial\Omega$ and some positive constant C. Here

$$\delta(z) = \delta_\Omega(z) = \operatorname{dist}(z, \partial\Omega), \quad z \in \overline{\Omega}, \tag{3}$$

and

$$f_w = \sum_j f_{z_j} w_j, \quad f_{\bar{w}} = \sum_j f_{\bar{z}_j} \bar{w}_j,$$

so that

$$f_{w\bar{w}} = \sum_{j,k} f_{z_j \bar{z}_k} w_j \bar{w}_k.$$

Our first result is the following quantitative version of the Diederich-Fornæss theorem:

Theorem 1. *Assume that Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. Let $C > 0$ be such that* (2) *holds and $R > 0$ such that $\Omega \subset B(z_0, R)$ for some $z_0 \in \mathbb{C}^n$. Then $(1 + CR)^{-2}$ is a DF exponent for Ω.*

If the boundary is C^k then we obtain a C^k defining function with this exponent. Theorem 1 thus gives

$$\eta_k(\Omega) \geq \frac{1}{(1 + C_\Omega)^2}, \tag{4}$$

where

$$C_\Omega := \operatorname{diam} \Omega \inf C/2$$

and the infimum is being taken over any neighborhood of $\partial\Omega$ where δ is C^k and C that satisfies (2) there. Note that C_Ω is invariant under linear transformations. It is also clear that $C_\Omega = 0$ if Ω is strongly pseudoconvex.

Classical examples of domains with arbitrary small DF indices are the worm domains defined and investigated by Diederich-Fornæss in [3]. For $\mu > 0$ they are given by (see also [9]):

$$\Omega_\mu := \left\{ z \in \mathbb{C}^2 \colon \left| z_1 - e^{i \log |z_2|} \right|^2 < 1 - \eta(\log |z_2|) \right\}$$

where η is C^∞, nonnegative, convex on $\mathbb{R}$ and such that $\{\eta = 0\} = [-\mu, \mu]$. This ensures that Ω is bounded pseudoconvex in $\mathbb{C}^2$ with C^∞ boundary, see [9] for details. It was proved in [3] that for $k \geq 3$ we have $\eta_k(\Omega_\mu) \leq \pi/2\mu$. Liu [10] showed that in fact $\eta_k(\Omega_\mu) = \pi/(\pi + 2\mu)$ for $k \geq 3$. The reason why both papers had to assume that $k \geq 3$ is that if $\rho, \widetilde{\rho}$ are two C^k defining functions then $\widetilde{\rho}/\rho$ is of class C^{k-1} and this regularity cannot be improved in general and the methods require that is ratio is C^2.

We will improve these results to C^2 defining functions and even to more general psh ones using some pluripotential theory in worm domains. The concept of the DF exponent and index can be generalized to nonsmooth functions as follows. Recall that a function is psh if its complex Hessian is nonnegative in the weak sense. This implies that u is in particular upper semi-continuous. A negative psh function u in Ω vanishing on the boundary will be called a *psh defining function* for Ω. In this case we will say that b is a *psh DF exponent* for Ω if near the boundary

$$|u| \leq C\delta_\Omega^b \tag{5}$$

for some $C > 0$. It is clear that then (5) also holds for an arbitrary psh defining function for Ω which is maximal near the boundary, for example the pluricomplex Green function of Ω with some fixed pole. Therefore, b is a psh DF exponent if and only if (5) holds near the boundary for the pluricomplex Green function of Ω for some (equivalently, any) pole in Ω.

From [2] (or Theorem 1) it follows that for bounded pseudoconvex domains with C^2 boundary there are psh defining functions satisfying (5) and even the corresponding lower bound. This was used by Herbort [8] to prove that then the pluricomplex Green function converges locally uniformly to 0 as the pole converges to the boundary (see also [1]). This, together with corresponding quantitative estimates for the Green function lead to lower bounds for the Bergman distance in such domains, see [4] and [1]. It was later proved by Harrington [6] that such a psh defining function exists in bounded pseudoconvex domains with Lipschitz boundary. Using that one sees that estimates from [8] and [1] immediately generalize from C^2 to Lipschitz boundaries.

We will prove that for the worm domains the indices for psh defining functions and C^3 defining functions are the same:

Theorem 2. *If b is a psh DF exponent for Ω_μ then $b \leq \pi/(\pi + 2\mu)$.*

We do not know if the limit value can be attained. We conjecture that, similarly as for C^3 defining functions (see Theorem 7 below), this is not possible. Equivalently, this would mean that the Green function $G_{\Omega_\mu}(\cdot, w)$ does not satisfy (5) near the boundary for $b = \pi/(\pi + 2\mu)$ for some (any) $w \in \Omega_\mu$. In the proof of Theorem 2 below we only show it on a certain complex disk in Ω_μ.

More generally, we conjecture that for any bounded pseudoconvex Ω with C^k boundary, $k, 2, 3, \ldots, \infty$, if 1 is not a DF exponent then the set of DF exponents for C^k defining functions is an open interval. We expect it also for psh DF exponents.

Using Theorem 2 we now see that previously mentioned results from [3] and [10] also hold for C^2 defining functions:

Corollary 3. $\eta_2(\Omega_\mu) = \dfrac{\pi}{\pi + 2\mu}$.

Combining this with (4) we get

Corollary 4. $C_{\Omega_\mu} \geq \sqrt{1 + \dfrac{2\mu}{\pi}} - 1$.

It would be interesting to compute C_{Ω_μ} precisely.

We conjecture that, similarly as in the case of worm domains, for bounded pseudoconvex domains with smooth boundary the DF indices for smooth defining functions and for psh defining functions are always the same.

The paper is organized as follows. Theorem 1 is proved in Section 2. In Section 3 we analyze C^3 defining functions for worm domains, in particular we obtain a different, more direct proof of the formula for the DF index of worm domains from [10]. Finally, we prove Theorem 2 in Section 4.

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2. DF Exponents for Smooth Domains

In order to present a complete proof of the existence of DF exponents we start with a proof of the crucial estimate (2) repeating the arguments from [2]. Pseudoconvexity implies that $-\log\delta$ is psh, where δ is given by (3), that is

$$\delta\delta_{w\bar{w}} \le |\delta_w|^2 \tag{6}$$

near the boundary. For a fixed point in $\overline{\Omega}$ near $\partial\Omega$ and $w \in \mathbb{C}^n$ we write $w = w' + w''$, where w' is tangent to the level set of δ at this point and w'' is normal to it. By (6) we have $\delta_{w'\overline{w'}} \le 0$ and

$$\delta_{w\bar{w}} \le 2\mathrm{Re}\,\delta_{w'\overline{w''}} + \delta_{w''\overline{w''}} \le C_1|w''|\,|w|.$$

Since

$$|\delta_w| = |\delta_{w''}| \ge \frac{1}{C_2}|w''|,$$

we get (2).

Proof of Theorem 1. We may assume that $z_0 = 0$. Take $a, b > 0$ with $a > 0$, $0 < b < 1$. For the function

$$u := -e^{-a|z|^2}\delta^b$$

we can compute

$$u_w = \left[a\delta\overline{\langle z, w\rangle} - b\delta_w\right]e^{-a|z|^2}\delta^{b-1}$$

and, with C given by (2),

$$\begin{aligned} u_{w\bar{w}}e^{a|z|^2}\delta^{2-b} &= 2ab\delta\mathrm{Re}\,(\delta_w\langle z, w\rangle) + a\delta^2|w|^2 - b\delta\delta_{w\bar{w}} - a^2\delta^2|\langle z, w\rangle|^2 + b(1-b)|\delta_w|^2 \\ &\ge -2abR\delta|\delta_w||w| + a\delta^2|w|^2 - Cb\delta|\delta_w||w| - a^2R^2\delta^2|w|^2 + b(1-b)|\delta_w|^2 \\ &= a(1-aR^2)\delta^2|w|^2 - b(2aR + C)\delta|\delta_w||w| + b(1-b)|\delta_w|^2. \end{aligned}$$

This is nonnegative provided that

$$4a(1-b)(1-aR^2) - b(2aR+C)^2 \geq 0.$$

This means that for any a between 0 and $1/R^2$

$$b = \frac{4a(1-aR^2)}{4a(1+CR)+C^2}$$

is a DF exponent for Ω. The maximum is attained for

$$a = \frac{C}{2R(1+CR)},$$

then

$$b = \frac{1}{(1+CR)^2}.$$

□

3. C^3 Defining Functions in Worm Domains

In this section we study DF exponents for C^3 defining functions in the worm domain Ω_μ. Consider the special C^∞ defining function for Ω_μ:

$$\rho = \left|z_1 - e^{i\log|z_2|}\right|^2 - 1 + \eta(\log|z_2|) \tag{7}$$

If $\widetilde{\rho}$ is a C^3 defining function for Ω_μ then it is the form $\widetilde{\rho} = \widetilde{h}\rho$, where $\widetilde{h} \in C^2(\overline{\Omega})$, $\widetilde{h} > 0$. We want to analyze when $-(-\widetilde{\rho})^b$ is psh, that is when functions of the form $v = -h(-\rho)^b$ are psh for $h \in C^2(\overline{\Omega})$, $h > 0$.

We will work with variables $z = z_1$, $t = \log|z_2|$. Without loss of generality we may assume that h is radially symmetric in z_2, otherwise replace it with the average over circles $\{|z_2| = r\}$. We may write

$$\rho = |z - e^{it}|^2 - 1 + \eta(t) = |z|^2 - 2\mathrm{Re}\,(e^{it}\bar{z}) + \eta(t). \tag{8}$$

We then have

$$\begin{aligned} \rho_z &= \bar{z} - e^{-it}, \quad \rho_{z\bar{z}} = 1, \quad \rho_{zt} = ie^{-it}, \\ \rho_t &= 2\mathrm{Im}\,(e^{it}\bar{z}) + \eta', \quad \rho_{tt} = 2\mathrm{Re}\,(e^{it}\bar{z}) + \eta''. \end{aligned} \tag{9}$$

With $v = -h(-\rho)^b$ we can compute

$$\begin{aligned} v_{z\bar{z}} &= (-\rho)^{b-2}\left[-h_{z\bar{z}}\rho^2 - 2b\rho\mathrm{Re}\,(h_z\rho_{\bar{z}}) - bh\rho\rho_{z\bar{z}} + b(1-b)h|\rho_z|^2\right], \\ v_{zt} &= (-\rho)^{b-1}\left[\rho h_{zt} + bh_z\rho_t + bh_t\rho_z + bh\rho_{zt} - b(1-b)h\rho_z\frac{\rho_t}{\rho}\right], \\ v_{tt} &= (-\rho)^b\left[-h_{tt} - 2bh_t\frac{\rho_t}{\rho} - bh\frac{\rho_{tt}}{\rho} + b(1-b)h\frac{\rho_t^2}{\rho^2}\right]. \end{aligned} \tag{10}$$

Using (9) we see that $\rho_z \to -e^{-it}$, $\rho_t \to 0$ and $-\rho_{tt}/\rho = -(|z|^2 - \rho + \eta'')/\rho \to 1$ as a sequence of points from Ω_μ converges to $(0,t) \in \{0\} \times [-\mu, \mu] \subset \partial\Omega_\mu$. If we assume in addition that $\rho_t/\rho \to 0$ (for example for points of the form $(\varepsilon e^{it}, t)$, as in [3]) we will obtain

Proposition 5. *For $v = -h(-\rho)^b$, where $h \in C^2(\overline{\Omega}_\mu)$, $b > 0$, and ρ given by (8) we have*

$$\liminf \left[(v_{z\bar z}v_{tt} - |v_{zt}|^2)(-\rho)^{2-2b}\right] \leq b\left[-(1-b)hh_{tt} - bh_t^2 - b^2h^2\right] \tag{11}$$

for any sequence of points in Ω_μ converging to $(0,t) \in \partial\Omega_\mu$. □

In fact, using that the coefficient in the determinant at ρ_t^2/ρ^2, equal to

$$b(1-b)h(-\rho)^{2-b}v_{z\bar z} - b^2(1-b)^2h^2|\rho_z|^2 = b(1-b)h(-\rho)\big[bh + h_{z\bar z}\rho + 2b\mathrm{Re}\,(h_z\rho_{\bar z})\big],$$

is positive in Ω_μ near $[-\mu,\mu]\times\{0\}$ but converges to 0 there, one can show that we have equality in (11).

Now let h be independent of z. We assume that $0 < b \leq 1$ and work for $t \in [-\mu,\mu]$, so that $\eta = \eta' = \eta'' = 0$. Using (9) and (10) we get

$$\begin{aligned} v_{z\bar z}(-\rho)^{2-b}/bh &= -\rho\rho_{z\bar z} + (1-b)|\rho_z|^2 = -\rho + (1-b)(1-\rho) =: A \geq 0,\\ v_{zt}(-\rho)^{2-b}/b &= -\rho\rho_z h_t - \rho h\rho_{zt} + (1-b)h\rho_z\rho_t,\\ v_{tt}(-\rho)^{2-b} &= -\rho^2 h_{tt} - b\rho h\rho_{tt} - 2b\rho h_t\rho_t + b(1-b)h\rho_t^2. \end{aligned}$$

Since $|\rho_z|^2 = 1-\rho$, $2\mathrm{Re}\,(\rho_{zt}\rho_{\bar z}) = \rho_t$ and $|\rho_{zt}|^2 = 1$, we can compute that

$$\begin{aligned} |v_{zt}|^2(-\rho)^{2b-4}/b^2 = (1-b)h^2\rho_t^2 A - \rho h\rho_t h_t(2A+\rho)\\ + \rho^2h^2 + b^2(1-\rho)\rho^2h_t^2. \end{aligned}$$

Therefore,

$$(v_{z\bar z}v_{tt} - |v_{zt}|^2)(-\rho)^{2-2b}/b = -Ahh_{tt} - b(1-\rho)h_t^2 + bh^2\left(A\frac{\rho_{tt}}{-\rho} - 1\right) + bhh_t\rho_t.$$

Using the inequalities

$$\begin{aligned} -\rho_{tt}/\rho = (\rho - |z|^2)/\rho \geq 1,\\ A \geq (1-b)(1-\rho) \geq 1 - b(1-\rho) \end{aligned}$$

we obtain

Proposition 6. *For $v = -h(-\rho)^b$, where $h = h(t) \in C^2([-\mu,\mu])$, $0 < b \leq 1$ and ρ given by (8) we have*

$$(v_{z\bar z}v_{tt} - |v_{zt}|^2)(-\rho)^{2-2b}/b \geq (1-\rho)\left[-(1-b)hh_{tt} - bh_t^2 - b^2h^2\right] + bhh_t\rho_t$$

in

$$\{(z,t) \in \overline{\Omega}_\mu\colon\ -\mu \leq t \leq \mu\}.$$ □

We can give a prove of the following result essentially proved by Liu [10]:

Theorem 7. *For the worm domain Ω_μ the set of all DF exponents for C^3 defining functions in Ω_μ is the open interval* $(0, \pi/(\pi+2\mu))$.

Proof. First assume that b is the DF exponent for Ω_μ for a C^3 defining function. This means that for some $h \in C^2(\overline{\Omega}_\mu)$, $h > 0$, the function $v = -h(-\rho)^b$ is psh Ω_μ, where ρ is given by (7). We may assume that h is radially symmetric with respect to z_2 (otherwise replace it by the appriopriate average) and work in variables (z,t). By Proposition 5 on $\{0\} \times [-\mu, \mu]$ we have

$$-(1-b)hh_{tt} - bh_t^2 - b^2h^2 \geq 0. \tag{12}$$

This is impossible for $b = 1$, we may thus assume that $b < 1$. With $\chi = (\log h)_t/b$ we see that (12) is equivalent to

$$\chi_t \leq -\frac{b}{1-b}(1+\chi^2)$$

that is

$$(\tan^{-1}\chi)_t \leq -\frac{b}{1-b}.$$

Since $\tan^{-1}\chi\colon [-\mu,\mu] \to [-\pi/2, \pi/2]$, it follows that $\mu b/(1-b) \leq \pi/2$. To exclude the equality case, we see that then on $[-\mu, \mu]$ we would have $\chi = -\tan(at)$, where $a = b/(1-b) = \pi/2\mu$, and $h(0,t) = c\cos^{1/a}(at)$ for some $c > 0$. This would contradict the condition $h > 0$, hence $b < \pi/(\pi+2\mu)$. We have thus proved that if b is the DF exponent for Ω_μ then $b < \pi/(\pi+2\mu)$.

Now assume that $0 < b < \pi/(\pi+2\mu)$ and let $\widetilde{a}$ be such that $b/(1-b) =: a < \widetilde{a} < \pi/2\mu$. Using the fact that for $h(t) = \cos^{1/\widetilde{a}}(\widetilde{a}t)$ we have equality in (12), one can easily show that for $h(t) = \cos^{1/a}(\widetilde{a}t)$ we have strict inequality in (12) on $[-\mu,\mu]$. By Proposition 6 the function $v = -h(-\rho)^b$ is psh near $[-\mu,\mu]\times\{0\}$ and the regularized maximum $\widetilde{\rho}$ of $h^{1/b}\rho$, ρ and $-\varepsilon$ for small $\varepsilon > 0$ is a C^∞ defining function for Ω_μ such that $-(-\widetilde{\rho})^b$ is psh. □

4. Psh Defining Functions in Worm Domains

Proof of Theorem 2. As explained in the introduction, it is enough to analyze the pluricomplex Green function at some pole. Without loss of generality we may thus assume that u is the Green function for Ω_μ with pole at $(1,1)$. For $\alpha = 2\mu/\pi$ consider the complex disk in Ω_μ

$$\varphi(\zeta) = \left(e^{i(\alpha+1)\zeta}, e^{\alpha\zeta}\right).$$

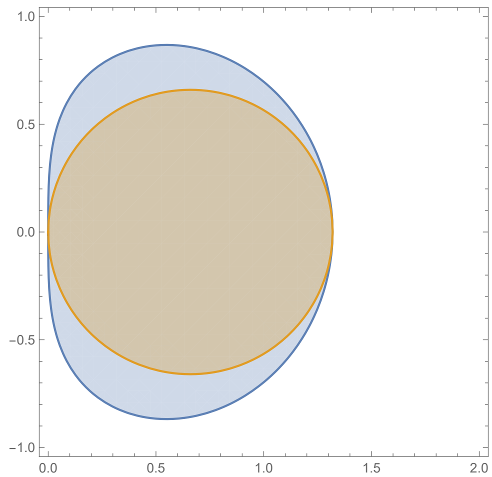
It is defined in

$$\begin{aligned} U &= \left\{\zeta \in \mathbb{C}\colon |\mathrm{Re}\,\zeta| < \pi/2,\ \left|e^{i(\alpha+1)\zeta - i\alpha\mathrm{Re}\,\zeta} - 1\right| < 1\right\} \\ &= \left\{\zeta \in \mathbb{C}\colon |\mathrm{Re}\,\zeta| < \pi/2,\ (\alpha+1)\mathrm{Im}\,\zeta > -\log\left(2\cos(\mathrm{Re}\,\zeta)\right)\right\}. \end{aligned}$$

We then have $u \circ \varphi \leq G_U(\cdot, 0)$, the Green function for U with pole at 0. The function $e^{i\zeta}$ biholomorphically maps the strip $\{|\mathrm{Re}\,\zeta| < \pi/2\}$ to the halfplane $\{\mathrm{Re}\, z > 0\}$. If $z = e^{i\zeta}$ then $|z| = e^{-\mathrm{Im}\,\zeta}$ and $\mathrm{Re}\, z = |z|\cos(\mathrm{Re}\,\zeta)$. Therefore

$$\{\mathrm{Re}\, z > 0\} \supset \{e^{i\zeta} \colon \zeta \in U\} = \{z \in \mathbb{C} \colon |z|^{\alpha+2} < 2\mathrm{Re}\, z\} \supset \{|z - \rho| < \rho\},$$

where $\rho = 2^{-\alpha/(\alpha+1)} > 1/2$ (see fig. below for $\alpha = 1.5$).



Comparing the Green functions at the origin for the preimages by the mapping $e^{i\zeta}$ of the half-plane and the inscribed disk we will obtain

$$\log\left|\frac{1 - e^{i\zeta}}{1 + e^{i\zeta}}\right| \leq G_U(\zeta, 0) \leq \log\left|\frac{1 - e^{i\zeta}}{1 + (1 - 1/\rho)e^{i\zeta}}\right|.$$

For $\zeta = iy$, $y > 0$, we get

$$u(e^{-(\alpha+1)y}, e^{i\alpha y}) \leq \log\left|\frac{1 - e^{-y}}{1 + (1 - 1/\rho)e^{-y}}\right|.$$

This means that for $t_k := e^{-2\pi k(\alpha+1)/\alpha} \to 0$ as $k \to \infty$

$$u(t_k, 1) \leq \log\left|\frac{1 - t_k^{1/(\alpha+1)}}{1 + (1 - 1/\rho)t_k^{1/(\alpha+1)}}\right|.$$

Therefore, if $|u| \leq C\delta_{\Omega_\mu}^b$ near the boundary then $b \leq 1/(\alpha + 1) = \pi/(\pi + 2\mu)$. $\square$

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